LINEAR AND NONLINEAR PROGRAMMING ON ANALOG COMPUTER

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
MASTER OF TECHNOLOGY

By
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to the

DEPARTMENT OF ELECTRICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

CONTENTS

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My acknowledgement is also due to Mr. S.V. Sirbhai for his efficient typing of the report.

August 1973. - M.L. MALIK

CERTIFICATE

This is to certify that the thesis entitled "LINEAR AND NON-LINEAR PROGRAMMING ON ANALOG COMPUTER" is a record of the work carried out under my supervision and that it has not been submitted elsewhere for a degree.

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ABSTRACT

The flexibility and ease of programming on an Analog Computer provide a theoretical-cum-experimental approach for the solution of most of dynamical problems and at the same time it provides a real insight into the behaviour of the system under study. A simple, fast and accurate method for solving linear and nonlinear programming problems using an Analog Computer is described in detail. The programming problem is interpretted as the asymptotic solution of a problem in the dynamics of a massless particle which is assumed to move in an n-dimensional space, away from those regions prohibited by the constraints of the problem and toward optimization of the objective function. The constraints or the objective function may not be linear but merely expressible with available Analog Computer components. A number of linear and nonlinear programming problems have been solved on three TR20 and the results compared with the solutions obtained using a digital computer. Post optimal analysis of these problems was also carried out.

CHAPTER I

1.1 Linear and Non-linear programming is concerned with finding optimal solutions to the static problems that arise in many fields of science and industry. The potential of an analog computer in solving such type of problems has often been overlooked partly because of the limited size of the problem that it can handle and partly because of limited accuracy. The size of the problem that can be solved on an Analog Computer is limited by the available number of computing amplifiers and the number of inputs to each amplifier. The latter difficulty is overcome by adding external equipment in the form of banks of coefficient potentiometers to the available Analog Computer TR-20. Regarding accuracy, however it must be borne in mind that a large number of industries employing optimization techniques are not able to provide problem data to the computer to an accuracy better than 1%. The analog computer method described obtain precision and accuracy well within the limitation imposed by the programming model itself.

A Digital Computer solves a programming problem in a step by step numerical calculation at discrete points whereas an Analog Computer will solve such a problem in a dynamic fashion even though the problem itself is static. The advantage of solving a programming problem in a dynamic fashion is speed, since the solution point is calculated continuously rather than discontinuously point by point, and the resulting solution even for a large programming problem takes only a matter of seconds.

Moreover the operation of the Analog Computer provides a very valuable insight into the nature of optimum seeking problem and its behaviour under varying parameters.

1.2 Outline of the thesis:

Chapter II outlines the theoretical background for solving programming problems on Analog Computers.

Chapter III explains the method of solving LP (Linear Programming) problem on TR-20 and the Design of external equipment to allow more number of inputs to each amplifier for medium sized LP problems.

Chapter IV illustrates a few LP problems and a transportation problem solution, along with their post-optimal analysis using TR-20 with additional inputs.

Chapter V explains the extension of analog computer method to nonlinear programming problems and illustrates some examples.

Chapter VI outlines other type of problems that can be solved using the same technique on Analog Computer.

Chapter VII is the conclusion.

CHAPTER II

THEORETICAL BACKGROUND

2.1 The general programming problem requires the evaluation of a number of unknowns so as to maximize or minimize an objective function subject to various constraints in the form of equalities and inequalities among functions of those unknowns. The programming problem may be expressed in the form:

Find x_i $(i = 1,2 \ldots\ldots n)$ to maximize

$$Z(x) = c_1 x_1 + c_2 x_2 + \ldots\ldots c_n x_n \tag{2.1}$$

subject to the constraints

$$A_j = \sum_{i=1}^{n} a_{ji} x_i \leq b_j \quad j = 1,2 \ldots\ldots\ldots m$$

$$\& \quad x_i \geqslant 0 \quad i = 1,2 \ldots n \tag{2.2}$$

If the coefficients c_1, C_2 .... c_n and a_{11}, a_{12} .... a_{mn} & b_1, b_2 ...... b_m are all constants, the resulting problem is a linear programming problem. Z and A_j may be arbitrary, single valued functions of x's. For the purpose of computation by the method to be described, the first derivative of $Z(x)$ must be defined throughout the solution space and the first derivative of A_j must be defined in the neighbourhood of the solution space boundaries. Electronic Analog computers are built to handle relations between variables and their time

rate of change. It is therefore required to prescribe the time rate of change of each variable appropriately. This time variation is in a fictitious analog time, for the problem being solved is a static one with no time dependence.

2.2 Geometrical Interpretation:

In geometrical language, the x_i are coordinates of particle Q in an n-dimensional space. The $\frac{dx_i}{dt}$ are components of velocity vector $\dot{Q}$ of the particle. The method of solution of programming problem on an analog computer is a variation of general method of steepest ascent (descent). The basic idea is to generate in the solution space a moving point Q that travels along the gradient of objective function to be maximized (minimized) until the optimum point is reached, thereby ensuring that the incremental changes in the solution displacement produce maximum change in the objective function. The velocity vector of the moving solution point can be defined by

$$\dot{Q} = s \nabla Z - K \sum_{j=1}^{m} \delta_j \nabla A_j (A_j - b_j) \qquad (2.3)$$

$$\text{where} \quad \delta_j = \begin{cases} 1 & \text{if } A_j > b_j \\ 0 & \text{if } A_j \leq b_j \end{cases} \qquad (2.4)$$

∇Z denote the gradient of the objective function $Z(x_1, x_2 \ldots . x_n)$ and is defined as

$$\text{grad } Z = \nabla Z = \frac{\partial Z}{\partial x_1} i_1 + \frac{\partial Z}{\partial x_2} i_2 + \ldots . \frac{\partial Z}{\partial x_n} i_n$$

where $i_1, i_2 \ldots\ldots i_n$ are the unit vectors of the cordinate system along $x_1, x_2 \ldots\ldots x_n$ respectively. Similarly ∇A_j defined a vector normal to the boundary hypersurface $A_j = b_j$.

s is an arbitrary real number, which may be allowed to go to zero as the solution point is reached.

K is an arbitrary large real number (may be 10 or 100).

In words the velocity of Q is given as the velocity along the direction of the gradient of objective function minus the sum of velocities in the direction of decrease of each A_j so long as constraint A_j is not satisfied. The signs of equation (2.3) are reversed if it is desired to minimize the objective function. Hence so long as the point Q $(x_1, x_2 \ldots\ldots x_n)$ is inside the n dimensional space (i.e. all constraints satisfied) the velocity vector lies along the gradient of the objective function, thereby moving in the direction of steepest Ascent (Descent). As some of the constraints are violated at some finite distance beyond the boundary, the value of $A_j - b_j$ will be such that the term $-K\nabla A_j\ (A_j - b_j)$ in equation (2.3) will just cancel the velocity component normal to the boundary. After that the velocity vector will lie parallel to the hypersurface for finite positive value of $(A_j - b_j)$. At points in the neighbourhood of relative maxima (minima) of Z, $\dot{Q}$ will vanish. The value of constant s can then be made arbitrarily small, thereby reducing the quantity $(A_j - b_j)$ and the point where $\dot{Q} = 0$ will be found in or at the boundary of the solution space.

2.3 The Analog Set up :

The essential elements of Analog Computer needed to instrument the equations (2.3) and (2.4) are:

(a) Integrators for obtaining x_i by integration of $\frac{dx_i}{dt}$

(b) Summers to compute the constraints

(c) Decision elements to generate δ_j and to ensure that x_i are $\geqslant 0$. The most convenient way to perform this decision function is through the use of diode limiting circuits.

The general procedure then is to write the time rate of change of variables x_i as

$$\dot{x}_i = s \frac{\partial Z}{\partial x_i} - K \sum_{j=1}^{m} \frac{\partial A_j}{\partial x_i} e_j \qquad i = 1,2 \ldots n \qquad (2.5)$$

where $$e_j = \delta_j (A_j - b_j) \qquad j = 1,2 \ldots m \qquad (2.6)$$

Equations (2.5) and (2.6) represent the general programming problem set up. For equations (2.1) and (2.2) we can write

$$\dot{x}_i = s c_i - K \sum_{j=1}^{m} a_{ji} e_j \qquad i = 1,2 \ldots n \qquad (2.7)$$

where $$e_j = \delta_j \left(\sum_{i=1}^{n} a_{ji} x_i - b_j \right) \qquad j = 1, 2 \ldots m \qquad (2.8)$$

The resulting computer set up for equations (2.7) and (2.8)

is shown in Fig.2.1. s is an arbitrary real number which may be allowed to go to zero as the solution is reached. Constant K is made large usually 10 or 100, in order that even a small error voltage of the order of few millivolts is fairly effective in keeping the solution point just near the boundaries of the allowed region.

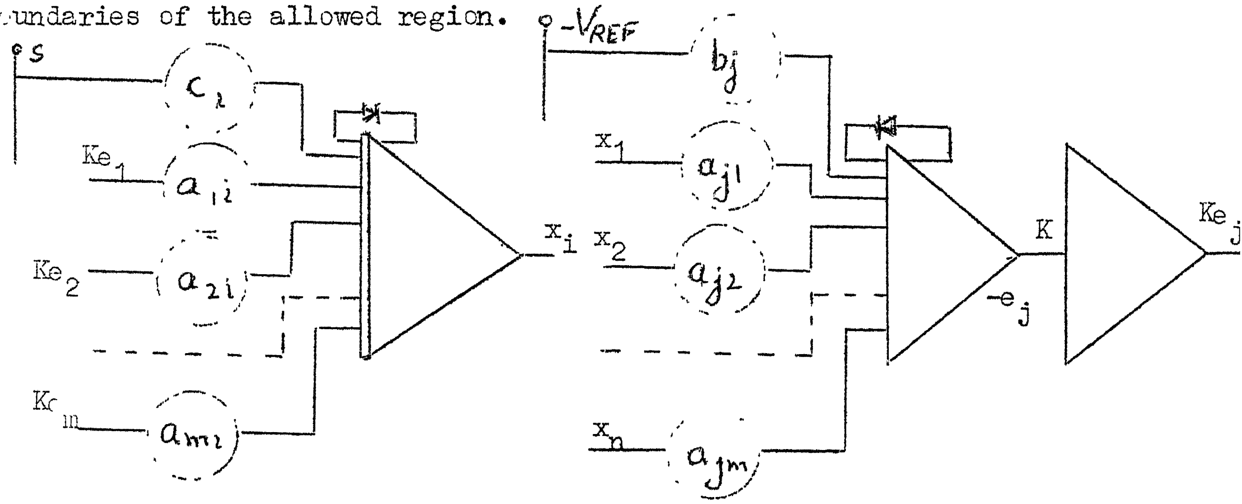

Fig.2.1: Computer Schematic for mathematical programming problem

The variables x_i are integrated with respect to time from the starting point determined by the initial conditions $x_1(0)$, $x_2(0)$ . . . . $x_n(0)$. The effect of error voltages Ke_1, Ke_2. . . . Ke_m fed back is to cancel the component of each $\dot{x}_i$ that is normal to the boundary hyper-surface, thereby constraining the point Q to move in the direction of increasing $Z(x)$ but parallel to the boundary surface. At this stage the point Q will lie a very small distance outside the allowed region for any finite value of s. For a finite value of s the velocity vector will always have a component along grad Z until maximum (minimum) value

of Z is reached, The value of s can then be made zero to get the solution point. The diode connected across the integrator in Fig.2.1 ensures that the variables x_i are always $\geqslant 0$ and the diode connected across the inverter shown in Fig.2.1 is used to keep the output of inverter as zero so long as $A_j \leq b_j$. When quantity A_j is larger than b_j, a voltage appears at the output of the summer. This voltage is multiplied by a gain constant K and fed back through an appropriate coefficient potentiometer to the integrator of the variable associated with that constraint. In addition to the elements shown, another summer is used to sum $c_1 x_1 + c_2 x_2 + \ldots \; c_n x_n = Z(x)$, the objective function.

2.4 Analytic Explanation:

The Analog Computer solution is based upon the property of an integrating amplifier that when the input voltage is made zero from a finite value, the output assumes a constant value. For maximization $-\frac{\partial Z}{\partial x_i}$ are fed to n integrators respectively so that when $Z(x_1, x_2 \ldots x_n)$ is driven to its maximum value, $\frac{\partial Z}{\partial x_i}$ have become zero, consequently the integrators input is zero and their outputs, now constant represent the value of x_i $(i = 1,2 \ldots n)$ which maximizes $Z(x_1, x_2 \ldots x_n)$.

The introduction of constraints into the optimization process is achieved by the use of error voltages e_j $(j = 1,2 \ldots m)$ such that e_j is a large negative voltage if jth constraint

$A_j = a_{j1}x_1 + a_{j2}x_2 + \ldots\ a_{jn}x_n \leq b_j$ is not satisfied and is zero if it is satisfied. The error voltage e_j is applied to the appropriate integrator x_i in the negative sense to decrease the value of x_i thereby readjusting the maximization process to satisfy the jth constraint. Diode feed back around the constraint **amplifier allow** either $e_j = 0$ (jth constraint satisfied) or e_j = large negative voltage (jth constraint not satisfied).

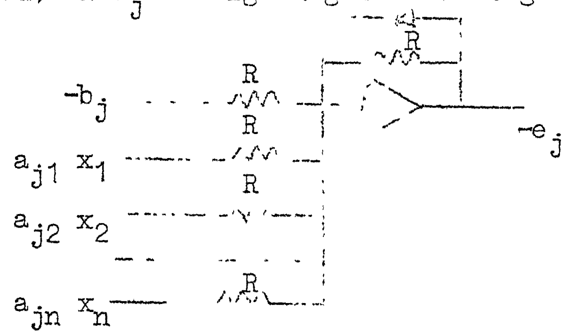

Fig.2.2: Error voltages e_j

The error voltage must be factored before feeding to the appropriate x_i integrator according to the effect the particular x_i has in generating the jth constraint i.e. for x_i, e_j is factored by a_{ji}. The condition $x_i \geq 0$ is ensured simply by diode feed back around x_i integrator as shown in Fig.2.3.

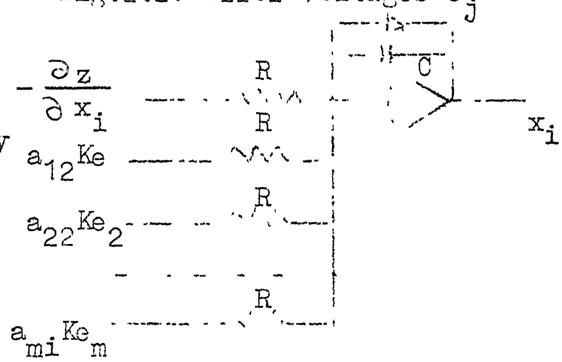

Fig.2.3; x_i integrator with constraint $x_i \geqslant 0$

2.5 Discussion:

It may not be possible to determine in advance whether a given programming problem has a solution or whether its solution is unique. Under these circumstances analog computer proves to be a handy exploratory tool. Any region of interest can be sufficiently

scanned by observing the solution with different sets of initial conditions thereby arriving at the desired solution. Scanning becomes necessary if solution point does not settle down to a point where all the constraints are satisfied. In case the solution point diverges toward infinity, the solution space is not closed in the direction of increasing values of the objective function, implying an improper programming problem.

2.5.1 LP Problems:

The method described in 2.3 works very well for LP problems since there is only one maximum (minimum) in the solution space. The solution will be obtained on the first trial starting from any initial point within a matter of seconds. The time being proportional to the value of s. All the functions appearing in LP problems are sums of constants or constant multiples of variables. Therefore an LP problem always leads to a connected convex set which will have only one point within the solution space (defined by the constraints) at which the objective function has maximum (minimum) value.

2.5.2 Nonlinear Programming Problems:

Nonlinear programming problems present two major difficulties:

(a) Describing the nonlinearities mathematically and taking these nonlinear functions into account during the set up. This presents no problem on Analog Computer because a non-linear

function is treated in the same manner as a linear function. However expressing nonlinear function on Analog Computer does require more elaborate equipment.

(b) Unlike LP problem, a nonlinear programming problem may have a number of local optimum points. This will warrant the scanning of the solution space by using different sets of initial conditions for each trial. Since a very large number of trials can be carried out in a relatively short time, this results in a good probability of finding the correct solution.

2.5.3 Sensitivity Analysis:

It is quite easy to change the coefficients in the objective function, the coefficients in the constraints or the constraints themselves, merely by resetting the appropriate potentiometers. Therefore one can very conveniently explore the sensitivity of the problem as a function of parameters of the system and/or find the effect on the system of changes predicted in future. However the use of Analog Computer has the disadvantage that large problems require a large amount of Analog equipment and obtainable accuracy decreases with increase in size of the problem.

CHAPTER III

3.1 A simple example will illustrate the Analog method of solving LP problem:

Example 1*

$$\text{Maximize} \quad Z = 3x_1 + 5x_2 \qquad (3.1)$$

subject to the constraints

$$\begin{aligned} x_1 \qquad\quad &\leq 4 \\ x_2 &\leq 6 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1 \geq 0 \; ; \quad x_2 &\geq 0 \end{aligned} \qquad (3.2)$$

Applying the method outlined in Chapter II the following time rates of change of two variables defined within the solution space are obtained:

$$\begin{aligned} \frac{\partial x_1}{\partial t} &= s\frac{\partial Z}{\partial x_1} = 3s \\ \frac{\partial x_2}{\partial t} &= s\frac{\partial Z}{\partial x_2} = 5s \end{aligned} \qquad (3.3)$$

The solution space is bounded by five lines (see Fig.3.1). If the solution point velocity vector is defined as above, the point will move along the path shown in dashed line until the first boundary A_3 is

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* Hillier, F.S., G.J. Lieberman "Introduction to Operation Research" pp.139

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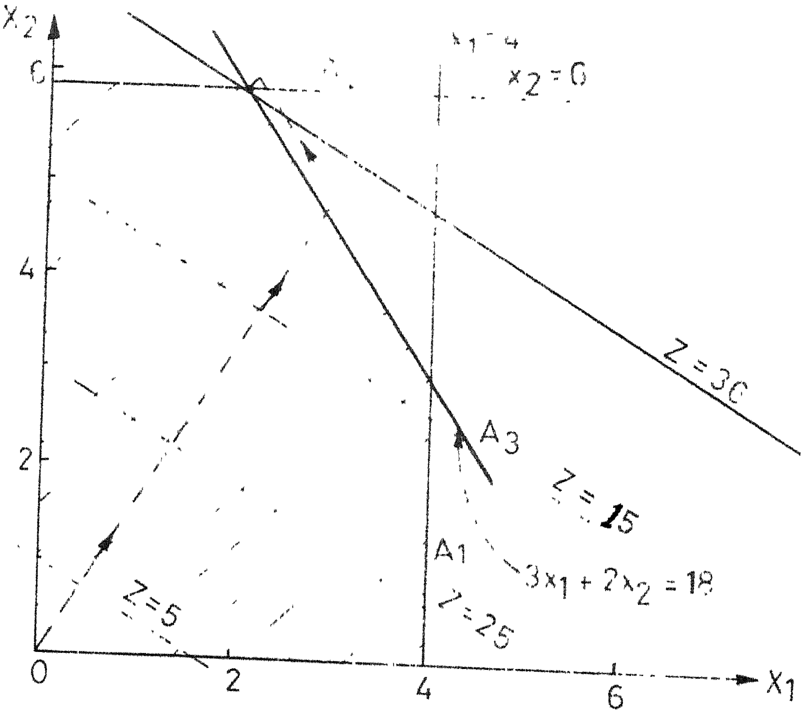

Fig. 3.1 Geometrical representation of example 1

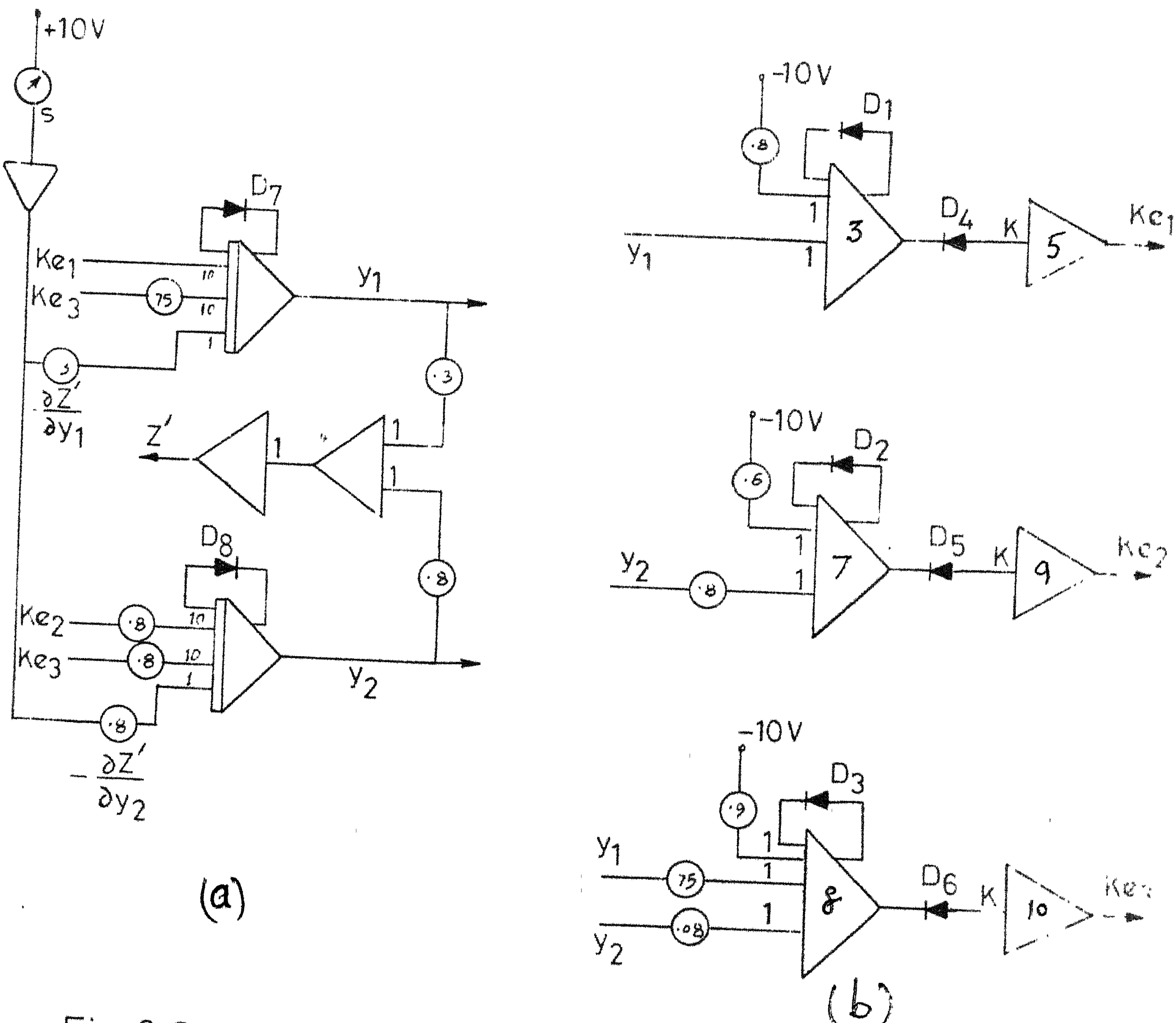

Fig. 3.2 Analog computer set-up for example 1

reached. If the component of the velocity vector perpendicular to A_3 is then removed by feed back circuits the point will continue to move along A_3 in the direction of increasing Z. When next constraint A_2 is reached, the solution point will cease to move and if now the value of s is allowed to approach zero the solution point will stop at the intersection of two constraints A_2 and A_3, giving the solution as Z=36.

3.1.1 Scaling:

It is obvious from equations (3.1) and (3.2) that scaling must be done first. In programming problems it is advisable to select scale factor for each variable to obtain the desired accuracy.

Let p_i $(i = 1,2 \ldots n)$ be the scaling factor for each $x_i (i = 1,2 \ldots n)$ respectively. Then

$$y_i = \frac{x_i}{p_i}$$ and substituting it in equations (3.2)

we get

$$\begin{aligned} p_1 y_1 \qquad\qquad & \leq 4 \\ p_2 y_2 \quad & \leq 6 \\ 3p_1 y_1 + 2p_2 y_2 & \leq 18 \end{aligned} \tag{3.4}$$

However it is obvious from equations (3.2) that x_1 and x_2 can at the most be equal to 4 and 6 respectively, therefore their scaling factor p_1 and p_2 should be 4 and 6 respectively. Allowing for some overshoot, devide each inequality of (3-4) by such a constant that makes each element of the RHS as less than or equal to 1. Therefore, dividing

equations srl. 1,2 and 3 of (3.4) by 5, 10 and 20 respectively, we get

$$\begin{aligned} \frac{p_1}{5}\, y_1 \qquad\qquad &\leq \quad 0.8 \\ \frac{p_2}{10}\, y_2 &\leq \quad 0.6 \\ \frac{3p_1}{20}\, y_1 + \frac{2p_2}{20}\, y_2 &\leq \quad 0.9 \end{aligned} \tag{3.5}$$

Now select the scale factors p_1, p_2 and p_3 such that all the constant elements of the constraints are less than or equal to 1.

Selecting $p_1 = 5$ $p_2 = 8$

and writing (3.5) in the table (3.1)

Constraint No.	Variables		Specification
	y_1	y_2	
1	1		$\leq$ 0.8
2		0.8	$\leq$ 0.6
3	0.75	0.8	$\leq$ 0.9

Table 3.1: Scaled coefficients of example 1

Now writing the objective function in terms of y_i

$$Z = 15\, y_1 + 40\, y_2$$

$$Z = \frac{Z}{50} = 0.3y_1 + 0.8\, y_2 \tag{3.6}$$

3.1.2 Analog Solution:

The analog computer circuit necessary to instrument table (3.1) and equation (3.6) is shown in Fig.(3.2).

The error voltages are set up from table 3.1 and are shown in Fig.3.2(b). Generation of y_1 and y_2 is indicated in Fig.3.2(a). Pot. (Potentiometer) settings of the scaled problem are indicated in the Fig.3.2 itself. Setting of pot s is initially unity. Germanium diodes were used. In the event of "constraint being satisfied", the error voltag were found to be small positive values of the order of 0.1 volts or so instead of being zero. Therefore intermediate diodes D4, D5 and D6 were added to achieve zero error voltage. The following procedure results in better accuracy than obtainable otherwise.

3.1.3 Operating procedure:

1. Patch up the various tested components of TR-20 as per the scaled diagram of the problem.

2. Set the various pots to the desired value, keeping the setting of pot s to unity. Pot setting of the external coefficient pots is done as per 3.4 and switch the computer from 'Reset' to 'operate'.

3. Allow 2-3 seconds to enable the outputs of various integrator/ amplifiers to attain steady state value.

4. Check the values of constraint amplifiers 3,7 and 8. For +ve error voltages make the corresponding factor K as 1 instead of 10. For

-ve error voltages short circuit the corresponding intermediate diode and keep K=10.

5. Reduce the setting of the pot s from unity to zero. Repeat steps 3 & 4 if found necessary.

The results obtained for example 1 are

$x_1 = .401 \times 5 = 2.005$ error $= \frac{.005}{2} \times 100 = .25\%$

$x_2 = .749 \times 8 = 5.992$ error $= \frac{.008}{6} \times 100 = .13\%$

$Z = .72 \times 50 = 36$

The above procedure ensures that all the constraints are being full satisfied and the results obtained are fairly accurate.

3.2 Additional aids to computation:

1. For minimizing instead of maximizing an objective function, apply $\frac{\partial Z'}{\partial y_i}$ instead of $-\frac{\partial Z'}{\partial y_i}$ to the y_i integrator.

2. To interchange the restricted and allowed region for any constraint edge, reverse the diode connected across the corresponding constraint amplifie

3. Negative coefficients among the constraints must be treated by the introduction of an inverter in the corresponding path since each such coefficient enters the wiring diagram twice. Two inverters are required for each negative term.

4. Negative coefficients in the objective function are handled by the application of $\frac{\partial Z'}{\partial y_i}$ instead of $-\frac{\partial Z'}{\partial y_i}$ at the appropriate integrator input.

5. To incorporate equality constraint instead of inequality constraint, remove the diode connected around the appropriate constraint amplifier.

6. If the integrating capacitors are all started from a discharged condition, the solution point will start its motion from the origin. For scanning purposes it may be desired to start the solution point from another initial condition. This is achieved by introducing appropriate initial voltages on the integrating capcitors.

3.3 Limitations of Analog Computer:

The main limitations in the use of Analog Cmputer for programming problems are:

1. Size of the problem is restricted by the available number of amplifier and potentiometers and the number of inputs to each amplifier.

2. The accuracy decreases with the increase in size of the problem.

The number of amplifiers required for any LP problem are $= n + 2m + p + 2$ out of which n are integrators

where n = Number of variables in the problem

m = Number of constraints in the problem

p = Number of rows plus columns containing negative coefficients in the LHS of constraints.

One amplifier is required to generate the objective function and another is required in generating $-\frac{\partial Z}{\partial y_i}$.

Number of potentiometers required for solving LP problem are $2g + 2h + m + 1$

where g = Total non zero coefficients on the LHS of the scaled constraints.

h = Number of non zero coefficients in the scaled objective function.

m = Number of scaled non zero RHS elements in the constraints.

Number of diodes required for solving LP problem are = n+2 x m

where n = Number of variables in the problem.

m = Number of inequality constraints in the problem.

Use of three TR-20 Analog computers provide 60 amplifiers (24 integrators) and 72 potentiometers. However, the main limitation comes from the limited number of inputs available to each amplifier. Since there are only two inputs of gain 1 and two inputs of gain 10 to each amplifier. It is obvious from the discussion so far that each constraint amplifier requires as many inputs of gain unity as there are number of coefficients in that constraint including RHS elements, which, for example may be 10 for a nine variable problem. Also each integrator requires as many inputs of gain 10 as number of coefficients in each column of set of constraints excluding RHS vector. It therefore warrants the design of an external equipment in the form of variable banks of extra inputs of gain 1 and gain 10 along with additic potentiometers to supplement the potential of the three TR 20 Analog Computers.

3.4 Design of External Equipment for additional inputs to TR-20:

The inputs resistors used in TR-20 are fixed Resistor, wire wound, precision type, ± .05% and of the values shown in Fig.3.3. The resistor selected for the external equipment are 10K and 100K, ± 1%, $\frac{1}{4}$W, metal film resistors which are highly stable. Only the resistors within ± .25% variation were selected for the purpose, To ease out

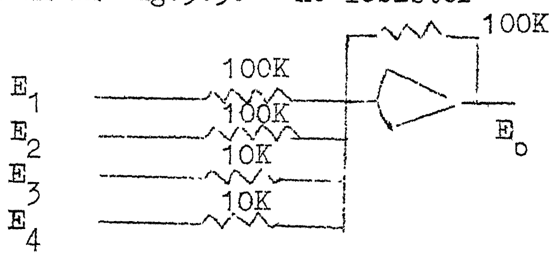

Fig.3.3: Input Resisters with each amplifier

the problem of additional pots, a 3.3K potentiometer was connected in series with each 10K/100K input resistor for various coefficient settings. The potentiometers used are single turn, 3.3K, metal film type which give fairly stable settings. The set up for this equipment is shown in Fig.3.4. The shaded circles represent the sockets for external connections. Suppose it is desired to add three additional inputs to any amplifier, the following procedure should be adopted in order to take into account the effect of loading and for accurate setting of external pots.

1. Interconnect the "To SJ" terminals of three rows (say A_1, A_2 and A_3) by jumper wires as shown in Fig.3.4.

2. Connect one of these "To SJ" terminals to the accessible SJ terminal of the amplifier whose number of inputs are required to be supplemented.

3. Feed -10 V at input terminal of A_1 and set the corresponding pot to the desired value by null method so that the output of the amplifier in question is same as desired pot setting. Remove -10V from input terminal of A.

4. Repeat step 3 for A_2 and A_3.

5. Connect the desired inputs to input terminals of A_1, A_2 and A_3.

Two external boards of the type shown in Fig.3.4 were constructed and fitted on top of two TR-20, thereby giving a total of 30 additional inputs of gain 1 and 30 additional inputs of gain 10. In addition 60 additional pots were also incorporated giving total number of available potentiometers for programming as 72 + 60 = 132. It is felt that this external equipment has greatly enhanced the capability of existing TR-20 Analog Computer in solving Linear and nonlinear programming problems.

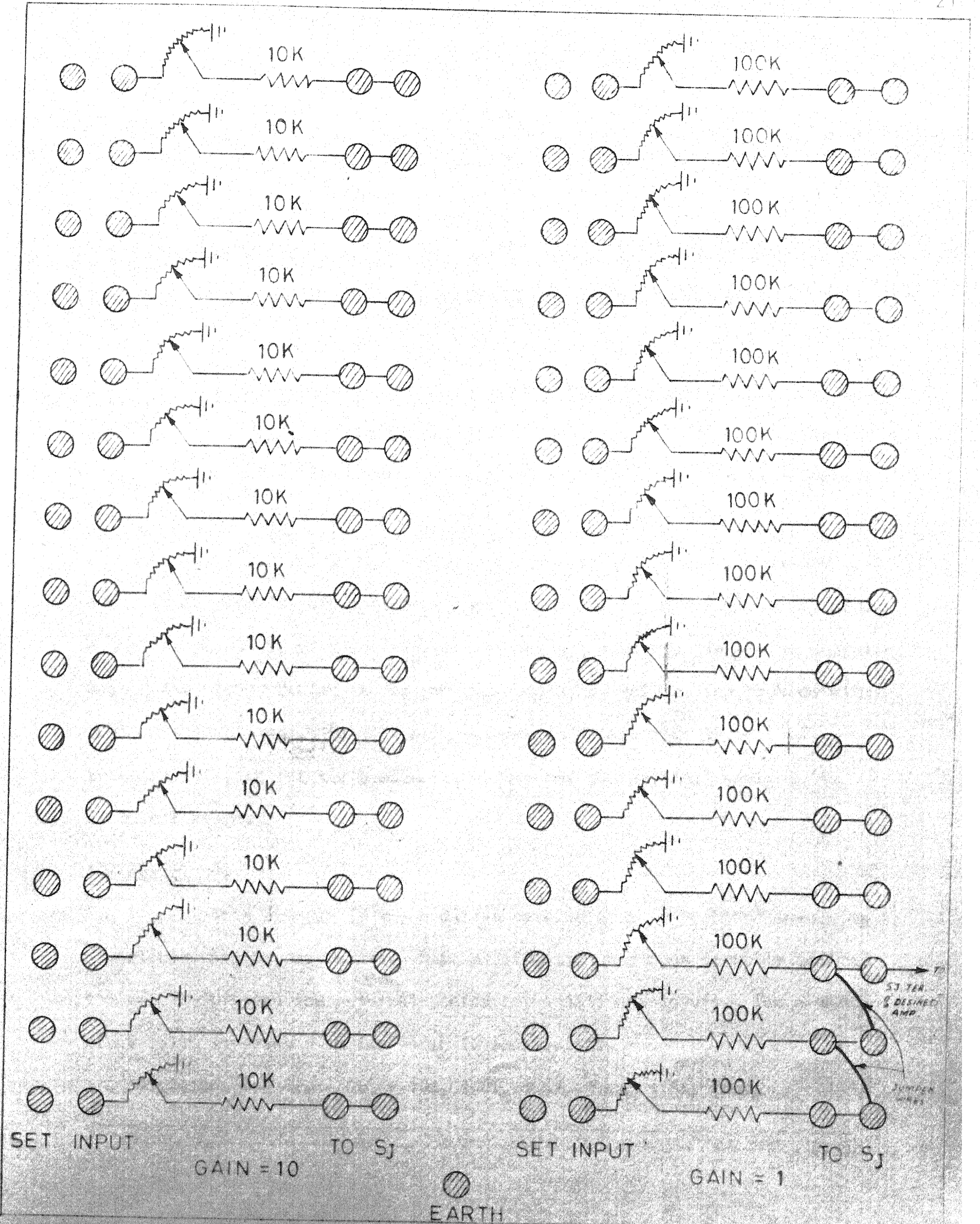

Fig.3.4 Enternal equipment to give additional inputs and pots for TR-20

CHAPTER IV

LP EXAMPLES WITH THEIR POST-OPTIMAL ANALYSIS

4.1 Introduction:

Once an LP problem of practical interest has been solved, situations may arise which require additional computation.

With practical problems we are often interested in not only the optimal solution of the given problem but also desire to know the effect of parameter changes on the solution, such as the profits (c_i) or the elements of requirement vector (b_j) or the elements of constraint vector (a_{ij}). In certain cases, after solving the problem, it may be discovered that one or more of the profits were incorrect, one or more of b_j were wrong or perhaps a decimal point was misplaced in some of the a_{ij} etc. etc. In Analog computer method the above problem can be overcome simply by solving the problem with revised pot settings. We will now consider in detail some typical LP problems bringing out the scaling procedure and post-optimal analysis as explained above.

4.2 Example 1:

This example is the modified version of a LP problem* involving 6 variables and 8 constraints. This modification was done to fully utilize the two TR-20's and the external boards for additional inputs. The example given below involves 8 variables in 12 constraints.

$$\text{Maximize} \quad Z = 8x_1 + 6x_2 + 10x_3 + 8x_4 + 5x_5 + 4x_6 + 8x_7 + 6x_8 \qquad (4.1)$$

*Hillier, F.S., G.S. Lieberman, "Introduction to Operation Research, pp.551 Ex.

Subject to:

$$
\begin{aligned}
&x_2 + x_3 + x_4 + x_5 + x_8 &\leq\ & 25 \\
&2.5x_1 + 2x_2 + x_3 + 2.5x_8 &\leq\ & 35 \\
&x_1 + 3x_2 + x_3 + x_7 &\leq\ & 15 \\
&2x_1 + x_2 + 4x_5 + x_8 &\leq\ & 15 \\
&x_3 + x_4 + x_6 &\leq\ & 15 \\
&2x_1 + 2x_5 + 3x_6 &\leq\ & 20 \\
&x_1 + x_3 + x_4 &\leq\ & 10 \\
&x_4 + x_5 + x_6 &\leq\ & 5 \\
&3x_6 + 5x_7 + 4x_8 &\leq\ & 15 \\
&3x_2 + 4x_4 + x_6 &\leq\ & 10 \\
&4x_3 + 3x_5 + x_7 &\leq\ & 20 \\
&5x_1 + x_7 &\leq\ & 35 \\
\text{and}\quad & x_i \geq 0 \text{ for } i = 1,2 \ldots . & & 8
\end{aligned}
\tag{4.2}
$$

4.2.1 Scaling: Because of the uniformity in the magnitude of the elements of various vectors, the scaling is simplified considerably. Dividing constrain (4.2) by 10 and normalizing all variables to one maximum value $x_m = 5$ such that $y_i = \frac{x_i}{5}$ for i = 1, 2 ...8. The scaled problem constraints are tabulated in table 4.1.

Writing objective function Z in terms of y_i and dividing it by 200, we get

$$\frac{Z}{200} = Z' = .2y_1 + .15y_2 + .25y_3 + .2y_4 + .125y_5 + .1y_6 + .2y_7 + .15y_8 \tag{4.3}$$

Constraint No.	Variables								Specification
	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	
1		.1	.1	.1	.1			.1	$\leq$ .5
2	.25	.2	.1					.25	$\leq$ .7
3	.1	.3	.1				.1		$\leq$ .3
4	.2	.1			.4			.1	$\leq$ .3
5			.1	.1		.1			$\leq$ .3
6	.2				.2	.3			$\leq$ .4
7	.1		.1	.1					$\leq$ .2
8				.1	.1	.1			$\leq$ .1
9						.3	.5	.4	$\leq$ .3
10		.3		.4		.1			$\leq$ .2
11			.4		.3		.1		$\leq$ .4
12	.5						.1		$\leq$ .7

Table 4.1: Coefficients of the scaled constraints for example 2

4.2.2 Analog Solution:

The analog computer diagram with the pot. settings is shown in Appendix 'A'. For this problem two TR-20's and 55 external inputs were utilized. The problem utilized the following analog computer components (refer to 3.3).

(a) Amplifiers = 8 + 2x12 + 2 = 34 (8 integrators)

(b) Potentiometers = 2x40 +2x8 + 12 + 1 = 109

(c) Diodes = 8 + 2x12 = 32

Using the procedure outlined in 3.1.3, the solution obtained is tabulated in table 4.2. The same problem was also solved on IBM 7044 using LP package, the results of which are also tabulated in table 4.2 for comparison sake.

Variables	Scaled solution	Analog solution	Digital solution	Error
x_1	.824	4.12	4.1704	1.2%
x_2	.405	2.025	2.0000	1.25%
x_3	.978	4.89	4.8295	1.24%
x_4	.203	1.015	1.0000	1.5%
x_5	.047	.235	0.2272	3.42%
x_6	0	0	0	-
x_7	0	0	0	-
x_8	.76	3.8	3.7500	1.33%
Z	.63	126	125.295	0.56%

Table 4.2: Comparison of Analog and Digital solutions for example 2

Error column in Table 4.2 represents the deviation of analog solution from digital solution expressed as percentage of the digital solution. It is seen that the error in x_5 is more pronounced than the other variables. This is attributed to the small magnitude of scaled solution. The accuracy

can be improved by using a smaller scaling factor for x_5.

4.2.3 Post optimal Analysis:

The effect of variation of RHS elements (Resources available) on the objective function was studied at discrete points. One element of the RHS vector (b_j) was varied at a time with the help of caliberated potentiometer and the corresponding value of the objective function recorded. The values of the variables (x_1, x_2 . . . x_8) were observed simultaneously and any changes from the basic feasible solution recorded. Table 4.3 gives the effect of variation of b_j on objective function. The corresponding curves are plotted in Fig.4.1 (a) and (b). Consider the variation of b_3 and its effect on objective function from table 4.3 and Fig.4.1(a). For $b_3 = 15$, $Z = 126$ i.e. optimum solution. This is represented by point A in Fig.4.1(a). As b_3 is decreased from 15, Z also decreases and when $Z = 112$, variable x_2 which was previously in the solution basis becomes zero, and remains so subsequently. This is indicate in the remarks column of table 4.3. A further decrease in b_3 reduces the val of objective function and other variables in the basis. If however after obtaining the optimum solution with given constraints, b_3 is increased from 15, Z also increases thereby indicating the room for further improvement in the objective function. It can be seen that Z can be improved only by increasing b_3, b_7, b_9, b_{10} or b_{11}. This was confirmed from the subroutine "RHS ranges" of the LP package. This subroutine gives only the upper and lower limit for each b_j so that the solution basis do not change. It is clear that the post optimal analysis on Analog

b_1	Z	Remarks	b_2	Z	Remarks	b_3	Z	Remarks	b_4	Z	Remarks	b_5	Z	Remarks	b_6	Z	Remarks
0	79.4	$x_8=0$, $x_2=0$	0	60.3	$x_8=0$	0	59.0		0	86.5		0	60		0	97.8	
2.5	104.6		5	63.7		5	95		3.75	90.7	$x_6>0$, $x_7>0$	2.5	84.8		2.5	107.4	
5	117.2	$x_5=0$, $x_4=0$	10	68.7	$x_7>0$	7	112	$x_2=0$	10	113.3		3.5	104.5		5	117.4	
7.5	121.2	$x_7>0$	11.25	71.3	$x_1=0$	10	117.2		13.5	123	$x_5=0$	5	122.4		6.25	121.8	$x_5=0$
10	123.1		15	84.3		15	126	optimum	15	126	optimum	10	126		10	124.5	
15	124	$x_6>0$	20	104.5		17	127	$x_5=0$	20	126		15	126	optimum	15	125.3	
20	125		25	116.9		20	128.6	$x_4=0$	25	126		20	126		20	126	
25	126	optimum	30	124.5		25	128.6		30	126		25	126		25	126	
30	126		35	126	optimum	30	128.6		35	126		30	126		30	126	
35	126		40	126		35	128.6		40	126							

Table 4.3: Record of the effect of variation of b_j on objective function (Z)

b_7	Z	Remarks	b_8	Z	Remarks	b_9	Z	Remarks	b_{10}	Z	Remarks	b_{11}	Z	Remarks	b_{12}	Z	Remarks
0	55.8		0	126	$x_5=0$	0	102.3	$x_8=0$	0	116.6	$x_7>0$	0	86.6	$x_3=0$	0	89.3	
1.25	66.8	$x_7>0$	5	126		5	110.3		3.75	121.6	$x_4=0$	5	99.6	$x_7>0$	2.5	93.3	
2.5	77	$x_1=0$	10	126	$x_6>0$	10	118.1		6	123.25	$x_5=0$	10	111		5	97.6	
5	93	$x_4=0$	15	126		15	126	Opt.	10	126	Opt.	15	121.8		10	106.8	
7.5	109.5		20	126		20	133.5	$x_5=0$	15	127.6	$x_6>0$	17.5	125.1	$\begin{cases} x_5=0 \\ x_6>0 \end{cases}$	15	115.5	
1[illegible]	126	Opt.				25	138.8	$x_6>0$	20	128.9		20	126	Opt.	20	124.4	
11	132	$x_5=0$				30	142.6	$x_7>0$	25	130.7	$x_6=0$	25	127.8	$x_6>0$	25	126	
12.5	137.4	$x_7>0$				35	147.6		30	132.9		27.5	128.6		30	126	
15	140.6					40	152.6		35	133.5		30	128.6		35	126	Opt.
20	141.8								40	133.5							
25	141.8																

Table 4.3: Record of the effect of variation of b_j on objective function (Z)

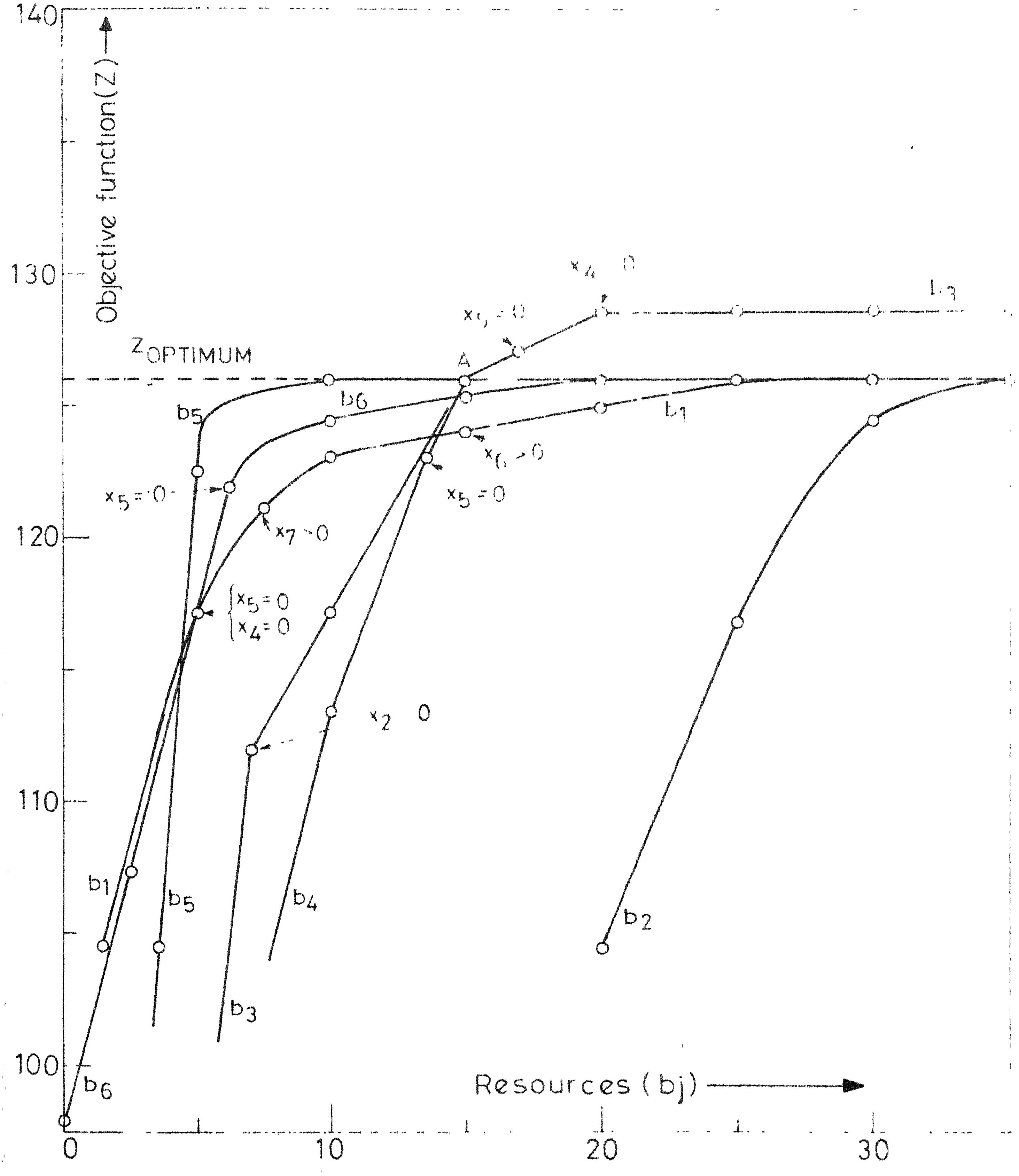

Fig.4.1 (a) Plots of objective function (Z) vs. resources (b_j) for example..

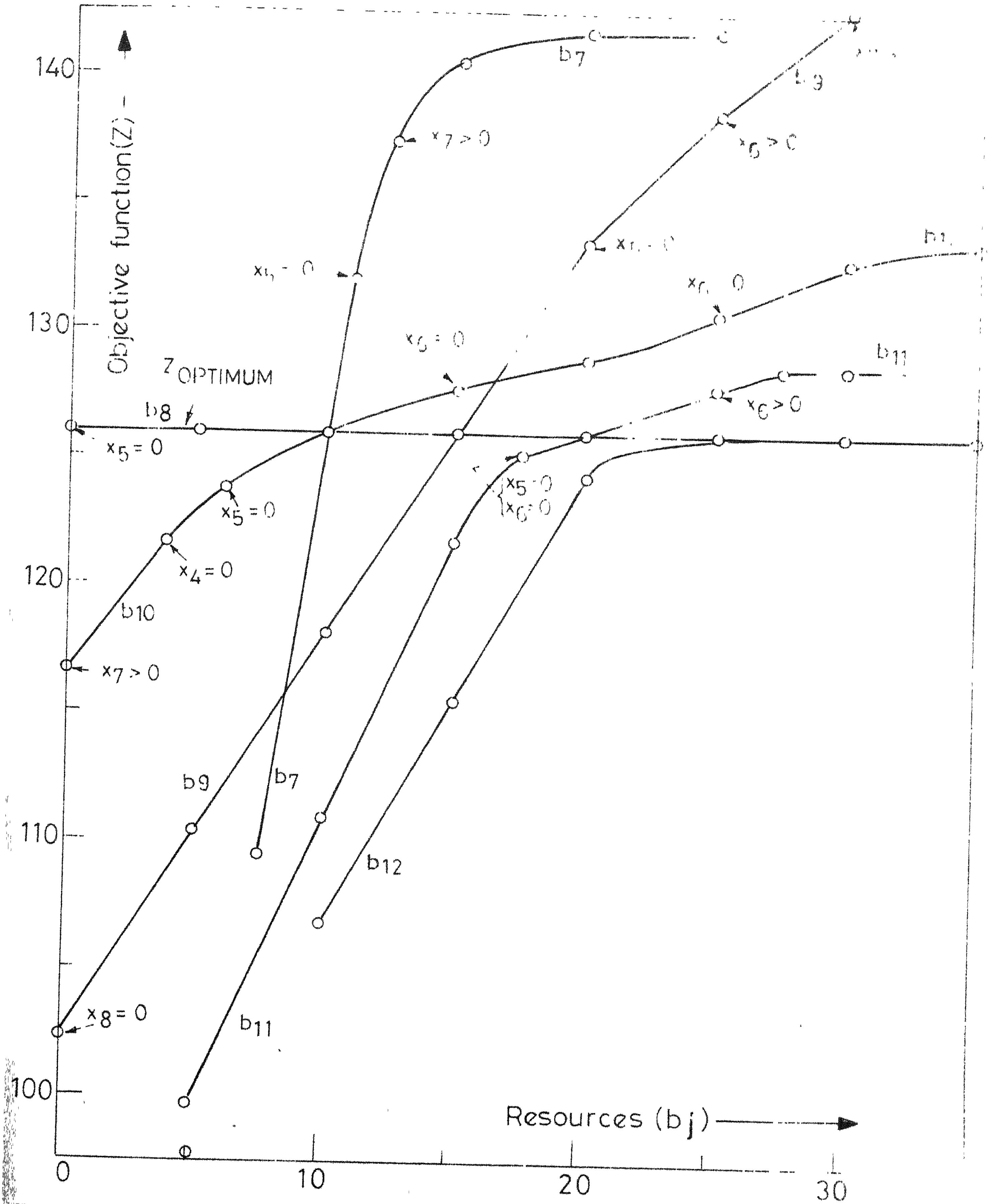

Fig. 4.1(b) Plots of objective function (Z) vs. resources (bj)

computer provides very valuable information easily whereas extensive digital computer programming will be required to achieve the same results.

4.3 Example 3:

This problem is the simplified version of an LP problem from a thesis work*. The size of the problem was cut down because the available number of analog computer components were not sufficient in number to solve it. This example involves 9 variables in 14 constraints. Unlike the last example where all the variables were scaled by the same factor, different scaling factors are chosen for different variables to give better accuracy.

Maximize

$$Z = 9.5\,x_1 + 12\,x_2 + 5.5\,x_3 + 15x_4 + 2x_5 + 5x_6 + 8x_7 + 10x_8 + 3.5x_9 \quad (4.4)$$

Subject to:

Multiplying factor			
1/3	$x_1 + x_2 + x_3$	$\leq$	2.75
1	$x_4 + x_5 + x_6$	$\leq$	.85
1	$x_2 + x_4$	$\leq$	.75
1/3	$x_1 + x_3 + x_5$	$\leq$	2.2
1	$x_7 + x_8$	$\leq$	.75
1/2	x_9	$\leq$	1.5
1/3	$.4x_3 + .7x_4 + .7x_6$	$\leq$	2.3
1/2	$.4x_1 + .25x_6 + 6.5x_8$	$\leq$	1.25

---

* Maj. A.V. Rao, "Algorithms for LP solutions for varying model Parameters", Thesis E.E., IIT-Kanpur.

Multiplying factor			
1/2	$.25x_7+2.2x_8+.1x_9$	≤ 1.2	
1/10	$.8x_1+2x_7+2.5x_8$	≤ 8.5	
1/10	$.3x_1+.5x_3+.5x_7+.5x_9$	≤ 8.5	
1/10	$1.2x_1+1.5x_3+1.5x_4+1.7x_6$	≤ 8.5	(4.5)
1/5	$1.3x_1+.15x_4+.15x_6+.9x_7+.3x_9$	≤ 2.5	
1/2	$.2x_2+6x_8$	≤ 2	
	& $x_i \geq 0$ for $i = 1,2 \ldots 9$		

4.3.1 Scaling: We write $y_i = \frac{x_i}{p_i}$ and multiply each constraint in (4.5) by such a constant so that each element in the RHS vector is less than or equal to unity. The values of the constants are indicated in the "multiplying factor" column shown in (4.5). From set of constraints (4.6), select constants p_i ($i = 1,2 \ldots 9$) corresponding to each y_i ($i=1,2\ldots9$) in such a way that all the LHS constraint elements of (4.6) are less than or equal to unity.

It can be visualized that non uniform scaling of the variables becomes quite involved, since all the constraints have to be checked for each p_i. It is therefore felt that, initially, all the variables should be scaled by the same constant and solution obtained on the analog computer. The solution itself shall warrant the rescaling of some of the variables which can be accomplished easily.

$$\frac{p_1}{3} y_1 + \frac{p_2}{3} y_2 + \frac{p_3}{3} y_3 \leq .9167$$

$$p_4 y_4 + p_5 y_5 + p_6 y_6 \leq .85$$

$$p_2 y_2 + p_4 y_4 \leq .75$$

$$\frac{p_1}{3} y_1 + \frac{p_3}{3} y_3 + \frac{p_5}{3} y_5 \leq .733$$

$$p_7 y_7 + p_8 y_8 \leq .75$$

$$\frac{p_9}{2} y_9 \leq .75$$

$$\frac{.4p_3}{3} y_3 + \frac{.7p_4}{3} y_4 + \frac{.7p_6}{3} y_6 \leq .767 \qquad (4.6)$$

$$\frac{.4p_1}{2} y_1 + \frac{.25p_6}{2} y_6 + \frac{6.5p_8}{2} y_8 \leq .625$$

$$\frac{.25p_7}{2} y_7 + \frac{2.2p_8}{2} y_8 + \frac{.1p_9}{2} y_9 \leq .6$$

$$\frac{.8p_1}{10} y_1 + \frac{2p_7}{10} y_7 + \frac{2.5p_8}{10} y_8 \leq .85$$

$$\frac{.3p_1}{10} y_1 + \frac{.5p_3}{10} y_3 + \frac{.5p_7}{10} y_7 + \frac{.5p_9}{10} y_9 \leq .85$$

$$\frac{1.2p_1}{10} y_1 + \frac{1.5p_3}{10} y_3 + \frac{1.5p_4}{10} y_4 + \frac{1.7p_6}{10} y_6 \leq .85$$

$$\frac{1.3p_1}{5} y_1 + \frac{.15p_4}{5} y_4 + \frac{.15p_6}{5} y_6 + \frac{.9p_7}{5} y_7 + \frac{.3p_9}{5} y_9 \leq .5$$

$$\frac{.2p_2}{2} y_2 + \frac{6p_8}{2} y_8 \leq 1$$

Selecting $p_1 = p_3 = 3$

$$p_2 = p_4 = p_5 = p_6 = p_7 = 1 \qquad (4.7$$

$$p_8 = 1/4\ p_9 = 2$$

The scaled constraints are written in Table 4.4.

Constraint No.	Variables									Specification
	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
1	1	.333	1							$\leq$ .9167
2				1	1	1				$\leq$ .85
3		1		1						$\leq$ .75
4	1		1		.333					$\leq$ .733
5							1	.25		$\leq$ .75
6									1	$\leq$ .75
7			.4	.233		.233				$\leq$ .767
8						.125		.8125		$\leq$ .625
9							.125	.275	.1	$\leq$ .6
10	.24						.2	.0625		$\leq$ .85
11	.09		.15				.05		.1	$\leq$ .85
12	.36		.45	.15		.17				$\leq$ .85
13	.78			.03		.03	.18		.12	$\leq$ .5
14			.3					.75		$\leq$ 1

Table 4.4: Coefficients of the scaled constraints for example 3.

The objective function Z is written in terms of y_i and dividing it by 50.

$$\frac{Z}{50} = Z' = .57y_1 + .24y_2 + .33y_3 + .3y_4 + .04y_5 + .1y_6 + .16y_7 + .05y_8 + .14y_9 \tag{4.8}$$

4.3.2 Analog Solution: The Analog Computer set up diagram alongwith with pot. Settings for table 4.4 and equation 4.8 is shown in appendix 'B' The problem utilized the following components (para 3.3 refers)

(a) Amplifiers = 9 + 2x14+2 = 39

(b) Pots = 2x40+2x9+14+1= 113

(c) Diodes = 9+2x14 = 37

Using the procedure outlined in 3.13, an Analog computer solution was obtained and is tabulated in table 4.5. The same problem was solved on IBM 7044 using LP package, the results of which are also tabulated in table 4.5 for comparison sake. The maximum error is about 3% in x_6. Percentage error column represents the deviation of digital solution expressed as percentage of the digital solution.

Variables	Scaled solution	Analog solution	Digital solution	Percentage error
x_1	.339	1.017	1.0314	1.4
x_2	.54	.54	.5500	1.82
x_3	.392	1.176	1.1685	.705
x_4	.205	.205	.2000	2.5
x_5	0	0	0	-
x_6	.63	.63	.6500	3.08
x_7	.639	.639	.6460	1.08
x_8	.424	.106	.1038	2.12
x_9	.751	1.502	1.5000	.133
Z	.81	40.5	40.5336	.089

Table 4.5: Comparison of Analog and Digital solution for example 3.

4.3.3 Post-optimal Analysis: In example 2, the effect of variation of RHS elements on objective function was studied. Here the effect of variation of the profit vector on the objective function will be studied at discrete points for this problem. One element of C_i of profit vector was varied at a time with the help of calibrated potentiometer and the corresponding values of the objective function recorded in table 4.6. The values of the variables $(x_1, \ldots x_9)$ were observed simultaneously and changes from the basic feasible solution recorded in the remarks column of table 4.6. The corresponding curves are plotted in fig.4.2(a) & (b).

Consider the effect of variation of c_5 on objective function from table 4.6 and fig.4.2. For $c_5 = 2$, $Z = 40.5$ is the optimum solution. This is represented by point A in fig.4.2(a). As c_5 is decreased from 2 to zero, Z is not affected. As c_5 is increased from 2 to 5, still Z remains 40.5. This is quite obvious since $x_5 = 0$ in the above range. However at $c_5 = 5$, x_5 enters the solution basis therety resulting in increase in the value of the objective function.

Careful study of fig.4.2 will give a valuable insight in the behaviour of the variation of objective function, as profit elements (c_i) are changed and one can select most profitable stregegy if some of the c_i are to be varied. For example decrease in the value of c_4 from 15 to zero results in reducing the value of the objective function merely from 40.5 to 39.6.

c_1	Z	Remarks	c_2	Z	Remarks	c_3	Z	Remarks	c_4	Z	Remarks	c_5	Z	Remarks
0	36.7		0	22.8	$x_2=0$	0	34.27		0	39.6		0	40.5	
1.67	36.7		2.5	28.13		3.33	38		5	39.6	$x_4=0$	2	40.5	Opt.
3.33	36.7		5	34		5.5	40.5	Opt.	10	40		5	40.5	
5	36.7		7.5	35.53		7.5	41.2		15	40.5	Opt.	10	40.5	
6.66	36.7	$x_1=0$	10	39.57		10	46.68	$x_1=0$	18.7	42.3	$x_2=0$	13.8	40.75	$x_5>0$
8.33	39.2		12	40.5	Opt.	13.3	55.4		20	43.6		15	42.8	$x_4=0$
9.5	40.5	Opt.	15	42		15	59.7		25	47		17.5	47.2	$x_6=0$
10	41.25		20	44.75		16.67	63.75		30	50.25		20	49.75	
13.33	44.9		25	47.48					35	55.15		25	54.15	
15	46.7		30	49.4					40	59.75		30	59.15	
16.67	48.5		35	51.75					45	64.4		35	64.15	

Table 4.6: Record of the effect of variation of c_i $(i=1,2\ \ldots 5)$ on objective function (Z)

c_6	Z	Re-marks	c_7	Z	Re-marks	c_8	Z	Re-marks	c_9	Z	Remarks
0	39.2		0	35.33		0	39.43		0	35.25	
1.75	39.2	$x_6=0$	5	38.55		10	40.5	Opt.	2.5	39	
5	40.5	Opt.	8	40.5	Opt.	20	41.55		3.5	40.5	Opt.
10	42.7		10	41.85		40	41.65		5	42.85	
13.75	47.15	$x_4=0$	15	45.1		60	45.2		7.5	46.65	
15	48.27		20	48.4		80	48.95		10	50.6	
20	53.6		25	51.85		100	53.85		12.5	55.15	
25	59.4		30	55.85		120	58.85		15	59.35	
30	62.9		35	59.65		140	63.6		17.5	63.85	
35	68.25		40	63.35		160	67.35		20	68.35	

Table 4.6: Record of the effect of variation of c_i $(i = 6,7,8,9)$ on objective function (Z)

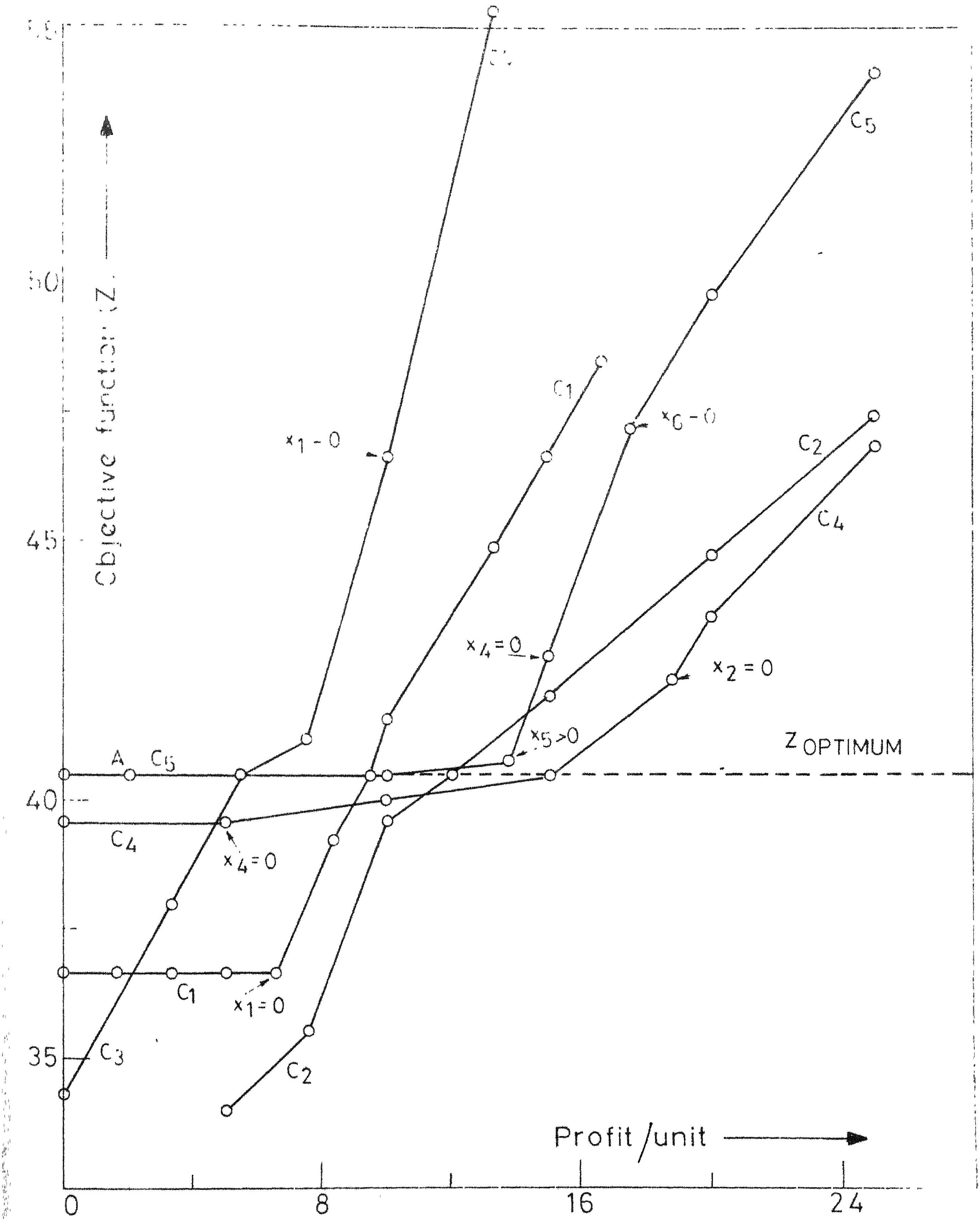

Fig.4.2 (a) Plots of objective function (Z) vs. profit elements (C_i)

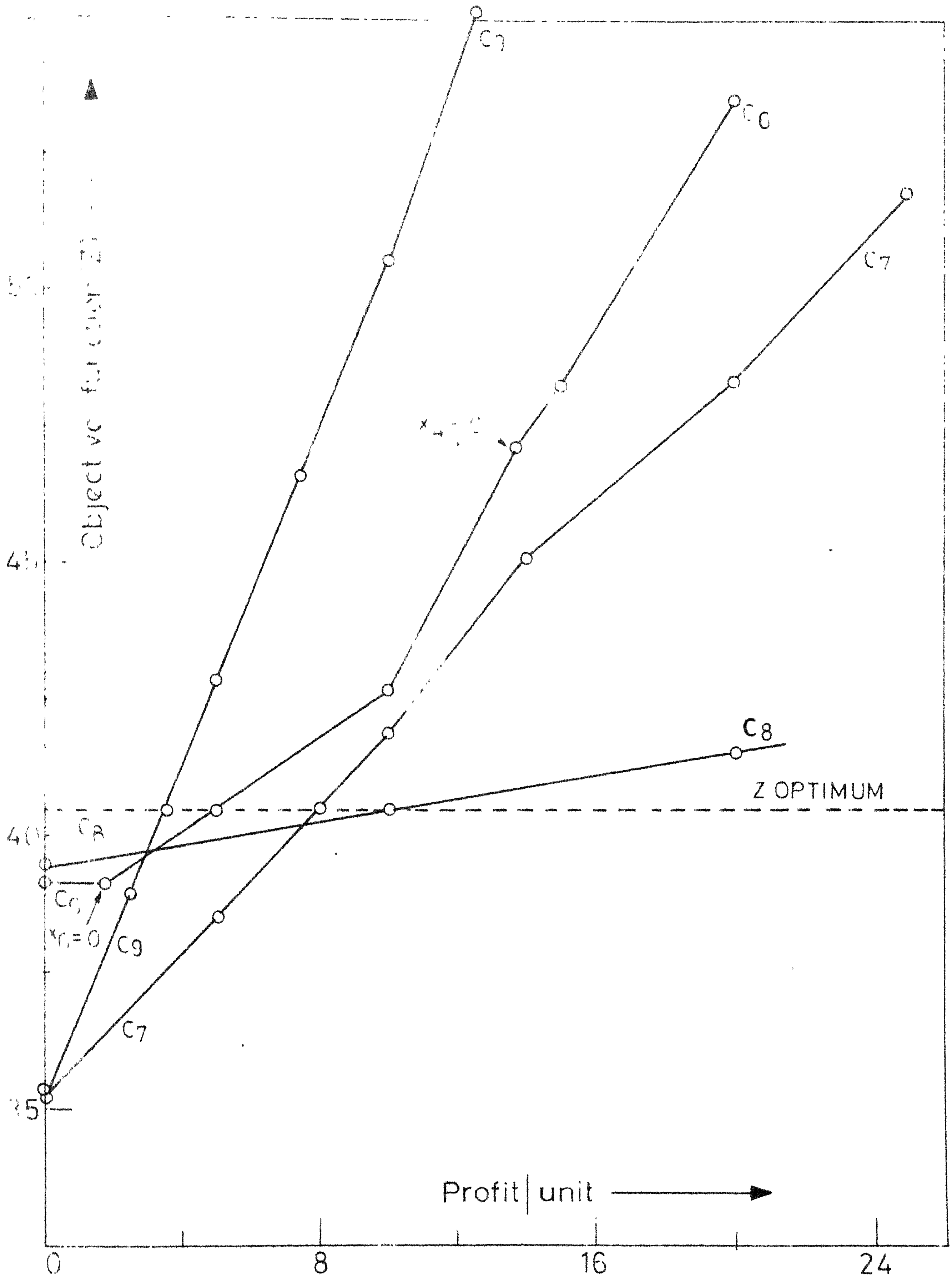

Fig.4.2(b) Plots of objective function vs. prof elements (Ci)

4.4 Example 4* This is a blending problem, the type of which occurs quite often in the petroleum industry. A refinery produces three grades of Aviation gasoline and automobile gasoline by blending four components (refinery streams) which are available in limited amount per day. There are two quality specifications for each aviation gasoline product.

(a) An upper limit on the admissible vapour pressure.

(b) A lower limit on the octane rating.

These two characteristics, which are measures of the volatality and the ignition properties of gasoline respectively, can be assumed to blend linearly. x_{ij} represent the amount of product j in barrels per day produced from the constituent i. The problem is to determine the optimal values of 16 unknowns x_{ij} subject to 10 restrictions, that can be derived from the data, namely, four material balances for the respective ingradients and two quality balances for each of three grades of aviation gasoline. Thus the objective function is to maximize.

$$Z = 5(x_{11} + x_{21} + x_{31} + x_{41}) + 5.5(x_{12} + x_{22} + x_{32} + x_{42}) + 6(x_{13} + x_{23} + x_{33} + x_{43}) + 4.5(x_{14} + x_{24} + x_{34} + x_{44}) \tag{4.9}$$

*Sven Dano, "Linear Programming in Industry", pp.45-46.

Subject to the following constraints:

$$
\begin{aligned}
&x_{11} + x_{12} + x_{13} + x_{14} = 3800\\
&x_{21} + x_{22} + x_{23} + x_{24} = 2650\\
&x_{31} + x_{32} + x_{33} + x_{34} = 4080\\
&x_{41} + x_{42} + x_{43} + x_{44} = 1300\\
&14x_{11} + 3x_{21} - 6x_{31} + 15x_{41} \geqslant 0\\
&2.0x_{11} - 1.0x_{21} + 3.0x_{31} - 13.5x_{41} \geqslant 0\\
&16.5x_{12} + 2x_{22} - 4x_{32} + 17x_{42} \geqslant 0\\
&2.0x_{12} - 1.0x_{22} + 3.0x_{32} - 13.5x_{42} \geqslant 0\\
&7.5x_{13} - 7x_{23} - 23x_{33} + 8x_{43} \geqslant 0\\
&2.0x_{13} - 1.0x_{23} + 3.0x_{33} - 13.5x_{43} \geqslant 0\\
&x_{ij} \geqslant 0 \text{ for } i = 1,2..4 \ \& \ j = 1,2..4
\end{aligned}
\tag{4.10}
$$

The RHS elements of first four constraints of (4.10) represent the amount of four constituents available in number of barrels per day.

4.4.1 Scaling: Dividing constraints Srl. 5 to 10 of (4.10) by 25 and writing $y_{ij} = \frac{x_{ij}}{5000}$ for $i = 1,2,3,4$

& $j = 1,2,3,4$

The scaled constraints are tabulated in table 4.7.

Writing the objective function Z in terms of y_{ij} and scaling, we get

$$
\begin{aligned}
\frac{Z}{5000x20} = Z' = {} & .25(y_{11}+y_{12}+y_{13}+y_{14})+.275(y_{21}+y_{22}+y_{23}+y_{24})\\
& + .3(y_{31}+y_{32}+y_{33}+y_{34}) + .225(y_{41}+y_{42}+y_{43}+y_{44})
\end{aligned}
\tag{4.11}
$$

Constraint No.	Variables																Specification
	y_{11}	y_{21}	y_{31}	y_{41}	y_{12}	y_{22}	y_{32}	y_{42}	y_{13}	y_{23}	y_{33}	y_{43}	y_{14}	y_{24}	y_{34}	y_{44}	
1	1				1				1				1				= .76
2		1				1				1				1			= .53
3			1				1				1				1		= .816
4				1				1				1				1	= .26
5	-.56	-.12	.24	-.6													$\leq$ 0
6	-.08	.04	-.12	.54													$\leq$ 0
7					-.66	-.08	.16	-.68									$\leq$ 0
8					-.08	.04	-.12	.14									$\leq$ 0
9									-.3	.28	.52	-.32					$\leq$ 0
10									-.08	.04	-.12	.552					$\leq$ 0

Table 4.7: Coefficients of the scaled constraints for example 4

4.4.2 Analog Solution: The Analog Computer set up diagram for table 4.7 and equations (4.11) along with the pot. settings is shown in Appendix 'C'. The problem utilizes the following components (Refer to para 3.3).

(a) Amplifiers = 16 + 2x10 + 2 = 38

(b) Pots = 2x24+2x16+4+1 = 85

(c) Diodes = 12 + 16 = 28

Using the procedure outlined in 3.13 the solution tabulated in table 4.8 was obtained. The same problem was solved on IBM 7044 using LP package, the results of which are also tabulated in table 4.8. for comparison sake. It is seen that % error in x_{21} and x_{44} is considerable.

Variables	Scaled Analog solution	Analog solution	Digital solution	Percentage error
x_{11}	0	0	0	-
x_{12}	0	0	0	-
x_{13}	0	0	0	-
x_{14}	0	0	0	-
x_{21}	.005	25	34.57	27.7
x_{22}	0	0	0	-
x_{23}	.591	2955	2958.73	.127
x_{24}	.134	670	662.62	1.12
x_{31}	.756	3780	3765.43	.39
x_{32}	.536	2680	2650	1.13
x_{33}	.221	1105	1121.26	1.45
x_{34}	.128	640	610.714	4.8
x_{41}	0	0	0	-
x_{42}	0	0	0	-
x_{43}	0	0	0	-
x_{44}	.0045	22	26.67	17.5
Z	.693	69300	69112	.272

Table:4.8: Comparison of Analog & Digital solution for example 4

The optimal values of x_{21} & x_{44} are 34.57 & 26.67 respectively and the scaling factor of 5000 used in scaling these variables is too large comparatively. This has resulted in very small scaled values of these variables and small error of the order of .001 in the scaled solution of these variables will reflect a larger percentage error. It is obvious that if the scaling factor for x_{21} & x_{44} could be made 100, the accuracy in the results of these variables will improve. However, the study of table 4.7 indicate that the scale factor for x_{21} & x_{44} can at the most be reduced to 1000 and 2500 respectively, without affecting the accuracy of other variables adversely. Therefore the accuracy for these variables could not be achieved better than 10% or so.

4.5 Example 5* (Transportation problem): One of the commonest application of Linear Programming is the transportation problem. Here it is required to arrange the flow of goods from say, a number of warehouses to the firm customers. The system cnstraints arise from the customer demand and availability of goods at each warehouse. The function to be minimized is the total transportation cost. A problem with three origions and seven destinations was studied using three TR-20 Analog Computers. Table 4.9 gives the transportation cost matrix (Rupees/Ton), the availability at three origins (Tons/day) and the demands (Tons/day) at seven destinations. The demand at each destination must be met.

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*Hadley, G.,"Linear Programming" pp.324, Prob.9.17.

Origins	Destinations D_1	D_2	D_3	D_4	D_5	D_6	D_7	Total availability Tons/day
O_1	10	8	16	3	10	25	18	10000
O_2	19	25	18	7	12	18	19	5000
O_3	20	17	20	5	14	16	17	10000
Total demand Tons/Day	2000	1000	3000	4500	500	600	950	

Table 4.9: Transportation cost matrix (Rupees/Ton), Availability (Tons/day) and demands (Tons/Day).

This problem can be expressed as the following linear programming problem.

$$\text{Minimize } Z = 10x_{11} + 8x_{12} + 16x_{13} + 3x_{14} + 10x_{15} + 25x_{16} + 18x_{17}$$
$$+ 19x_{21} + 25x_{22} + 18x_{23} + 7x_{24} + 12x_{25} + 18x_{26} + 19x_{27} \qquad (4.12)$$
$$+ 20x_{31} + 17x_{32} + 20x_{33} + 5x_{34} + 14x_{35} + 16x_{36} + 17x_{37}$$

Subject to the constraints

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} \leqslant 10000$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} + x_{27} \leqslant 5000$$
$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} + x_{37} \leqslant 10000$$
$$x_{11} + x_{21} + x_{31} = 2000$$
$$x_{12} + x_{22} + x_{32} = 1000$$
$$x_{13} + x_{23} + x_{33} = 3000$$

$$\begin{aligned} x_{14} + x_{24} + x_{34} &= 4500 \\ x_{15} + x_{25} + x_{35} &= 500 \\ x_{16} + x_{26} + x_{36} &= 600 \\ x_{17} + x_{27} + x_{37} &= 950 \end{aligned} \qquad (4.13)$$

& $x_{ij} \geqslant 0$ for $i = 1,2,3$ & $j = 1,2,\ldots.7$.

4.5.1 Scaling: This problem has 21 (3x7) variables and 10 (3+7) constraints. It is clear that the size of transportation problem, that can be solved on Analog Computer, is limited by the available number of integrators, which is only 24 for three TR-20 (8 integrators each). However, setting up of transportation problem is much less involved because of the unity coefficients in the LHS of constraints. This elliminates the patching and the setting up of number of pots , thereby saving considerable amount of time and effort. In this problem, to reduce the scaling factor to 5000 first and third constraints in (4.13) are divided by 2.

writing $y_{ij} = \frac{x_{ij}}{5000}$ for $i = 1,2,3$

& $j = 1,2,\ldots 7$

The coefficients of the scaled LP problem are tabulated in Table 4.10. Writing Z in terms of y_{ij} and scaling it by a factor of 5000x25.

$$\begin{aligned} \frac{Z}{5000x25} = Z' = {} & .4y_{11} + .32y_{12} + .64y_{13} + .12y_{14} + .4y_{15} + y_{16} + .72y_{17} \\ & + .76y_{21} + y_{22} + .72y_{23} + .28y_{24} + .48y_{25} + .78y_{26} + .76y_{27} \\ & + .8y_{31} + .68y_{32} + .8y_{33} + .2y_{34} + .56y_{35} + .64y_{36} + .68y_{37} \end{aligned} \qquad (4.14)$$

Cons-traint No.	Variables																					Speci-fication
	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}	y_{16}	y_{17}	y_{21}	y_{22}	y_{23}	y_{24}	y_{25}	y_{26}	y_{27}	y_{31}	y_{32}	y_{33}	y_{34}	y_{35}	y_{36}	y_{37}	
1	.5	.5	.5	.5	.5	.5	.5															$\leq$ 1
2								1	1	1	1	1	1	1								$\leq$ 1
3															.5	.5	.5	.5	.5	.5	.5	= 1
4	1							1							1							= .4
5		1							1							1						= .2
6			1							1							1					= .6
7				1							1							1				= .9
8					1							1							1			= .1
9						1							1							1		= .12
10							1							1							1	= .195

Table 4.10: Scaled constraint coefficients for example 4.

4.5.2 Analog Solution: The analog computer set up diagram for table 4.10 and equation (4.14) is shown in appendix 'D'. This problem utilized the following Analog Computer components (refer to 3.3).

(a) Amplifiers = 21 + 2x10 + 2 = 43

(b) Pots = 2x14 + 2x19 + 7+1 = 74

(c) Diodes = 21 + 2x3 = 27

Using the procedure outlined in 3.1.3 the optimal solution obtained is tabulated in table 4.11. The results obtained from IBM 7044 using LP package are also tabulated in the same table for comparison sake. It is seen that the maximum error is about 2%, which is quite acceptable

4.5.3 Post-Optimal Analysis: The effect of variation of amount available at each origin on the objective function was studied at discrete points and plotted as O_1, O_2 & O_3 in Fig.4.3. The demands (Tons/day) at each destination (D_1, D_2....D_7) was also varied with the help of calibrated potentiometer and its effect on the objective function observed. These observations are tabulated in table 4.12 and the corresponding curves plotted as shown in Fig.4.3. Consider the curve O_1 in fig.4.3, representing the quantity available in Tons/day at origin 1. For optimal value of Z = 121250, the availability desired at origin 1 (O_1) is 10,000 Tons/day. It is seen from table 4.12 and fig.4.3 that any decrease in O_1 from this value results in the increase in Z, thereby implying more cost of transportation. Whereas Z is not affected so long as O_2 and O_3 are greater than 1000 Tons/day and 1500 tons/day respectively. Study of demand curves (D_1, D_2...D_7) reveal

Variables	Scaled Analog Solution	Analog solution	Digital solution	Percentage error
x_{11}	.402	2010	2000	.5
x_{12}	.202	1010	1000	1
x_{13}	.397	1985	1999	.7
x_{14}	.896	4480	4500	.445
x_{15}	.105	505	500	1
x_{16}	0	0	0	-
x_{17}	0	0	0	-
x_{21}	0	0	0	-
x_{22}	0	0	0	-
x_{23}	.198	990	1001	1.1
x_{24}	0	0	0	-
x_{25}	0	0	0	-
x_{26}	0	0	0	-
x_{27}	0	0	0	-
x_{31}	0	0	0	-
x_{32}	0	0	0	-
x_{33}	0	0	0	-
x_{34}	0	0	0	-
x_{35}	0	0	0	-
x_{36}	.122	610	600	1.667
x_{37}	.186	930	950	2.1
Z	.97	121250	122250	.815

Table 4.11: Comparison of analog and digital solutions for example 5

O_1	Z	Remarks	O_2	Z	Remarks	O_3	Z	Remarks	D_1	Z	Remarks	D_2	Z	Remarks
0	138250	$x_{12} = 0$	0	123250	$x_{23}=0$	0	124250	$x_{36}=0$, $x_{37}=0$	0	99000		0	111500	
1000	135000	$x_{11}=0$	500	122250		1000	122250		500	104250		500	116000	
2000	133500	$x_{21}>0$	1000	121250	$x_{33}>0$	1500	121250	$x_{26}>0$, $x_{27}>0$	1000	109250	$x_{23}=0$	1000	121250	Opt.
3000	131750	$x_{14}=0$	2000	121250		2000	121250		2000	121250	Opt.	2000	131250	
4000	129500	$x_{24}>0$	3000	"		3000	"		3000	133250		3000	141000	$x_{13}=0$
5000	127750		4000	"		4000	"		3500	138750		3500	145250	$x_{25}>0$
6000	125750	$x_{34}>0$	5000	"	Opt.	5000	"		4000	145000	$x_{13}=0$	3750	148000	$x_{34}>0$
6500	125000	$x_{15}=0$				6000	"		4500	150750	$x_{25}>0$	4000	150500	$x_{15}>0$
7000	123750	$x_{25}>0$				7000	"		5000	156750	$x_{34}>0$	4500	155750	
8000	122500	$x_{13}=0$, $x_{32}>0$				8000	"					5000	160750	
10000	121250	Opt.				10000	"	Opt.						

Table 4.12: Record of effect of variation of availability/demand on objective function (Z)

D_3	Z	Remarks	D_4	Z	Remarks	D_5	Z	Remarks	D_6	Z	Remarks	D_7	Z	Remarks
0	63000	$x_{13}=0$	0	105750	$x_{14}=0$	0	115250	$x_{15}=0$	0	111500	$x_{36}=0$	0	105500	
500	73000		500	107125		500	121250	Opt.	600	121250	Opt.	500	113250	
1000	83000		1000	108500		1000	127250		1000	127500		950	121250	Opt.
2000	102500	$x_{23}=0$	2000	111500		2000	139500		2000	143500		1500	130000	
3000	121250	Opt.	3000	114500		2500	145500	$x_{13}=0$	3000	159000		2000	138500	
4000	140000		3500	116000	$x_{23}=0$	3125	152250	$x_{25}>0$	3500	167500		3000	155500	
5000	158000		4000	118500		3750	160250	$x_{34}>0$	4000	175500		4000	172250	
			4500	121250	Opt.	4000	163250		4500	183000		5000	188875	
			5000	123750		4500	169500		5000	191250				
						5000	175500							

Table 4.12: Record of effect of variation in availability/demand on objective function (Z)

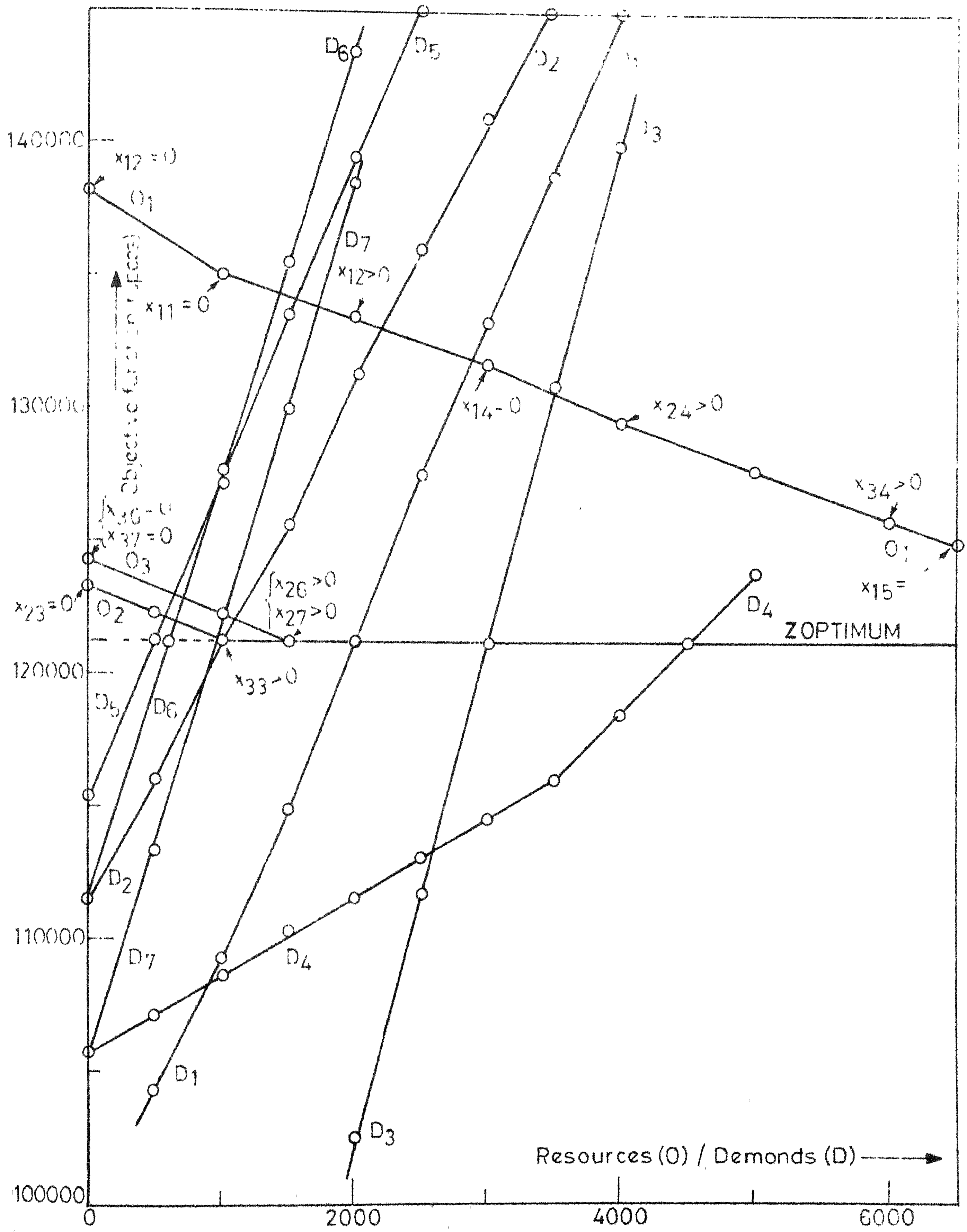

Fig.4.3 Plot of objective function vs. resources(O)/demonds (D)

that increase in demand results in increased transportation cost and decrease in demand reduces the transportation cost. It may be observed that the effect of increase in demand D_4 is less pronounced in increasing the transportation cost.

4.6 Discussion: It is seen that non-uniform scaling of the variables becomes quite involved as in example 3 while the uniform scaling may at times fail to achieve desired accuracy, as in example 4. It is now possible to offer some guidelines for the scaling procedure.

(a) To start with, scale all the variables of the given model by the same factor as in examples 2 and 4 and compute the results on the Analog Computer.

(b) In case the magnitudes of some of the variables are too small or beyond the range of Analog Computer, use appropriate scale factors for the affected variables and modify the scaled coefficients accordingly. Compute the solution with modified pot. settings.

It is clear from the four examples that medium sized linear programming problems having upto 15-20 variables/constraints can be solved on Analog Computer with a reasonable accuracy. The solution is obtained in a single run in a matter of few seconds and therefore it is a reasonably fast method of solving LP problems. However there is one drawback. When one of the constraint hyperplanes coincide with the hyperplane of the objective function, an infinite number of solutions can result. This situation can be checked by having 2-3 runs each with different initial

conditions. In LP package (IBM 7044), use of "RHS RANGING" & "COST RANGING" control cards will give the lower and the upper limit for $b_j(j=1,2\ldots m)$ & c_i $(i=1,2\ldots m)$ respectively for which the optimal solution basis remain unchanged. However to observe the variation in objective function with the varying parameters will involve extensive digital computer programming. In Analog method the variations in resources available or the variation in cost/profit coefficients can very easily be simulated by just resetting the relevant potentiometers. This gives a very valuable insight into the practical problems and here the Analog computer definitely has an edge over the digital computer.

CHAPTER V

NON-LINEAR PROGRAMMING

5.1 The general non-linear programming problem is to find $x_1, x_2 \ldots x_n$ so as to maximize $Z\ (x_1, x_2\ \ldots x_n)$ (5.1)

Subject to

$$\begin{aligned} A_1\ (x_1, x_2 \ldots\ldots x_n) &\leq b_1 \\ A_2\ (x_1, x_2 \ldots\ldots x_n) &\leq b_2 \\ \ldots\ldots\ldots\ldots \\ A_m\ (x_1, x_2 \ldots\ldots x_n) &\leq b_m \\ \& \ x_i \geqslant 0 \text{ for } i = 1,2 \ldots & \quad n \end{aligned} \tag{5.2}$$

where $Z\ (x_1, x_2 \ldots x_n)$ and $A_j\ (x_1, x_2 \ldots x_n)$ are given functions of n decision variables.

5.2 Analog Method of solution: The Analog method of solving LP problems can be extended to solve those non-linear programming problems which are expressible on analog computer. With slight modification writing equations (2.5) and (2.6) for general programming problems

$$\dot{x}_i = s_i \frac{\partial Z}{\partial x_i} - K \sum_{j=1}^{m} \frac{\partial A_j}{\partial x_i} e_j \quad \text{for } i = 1,2 \ldots n \tag{5.3}$$

$$\text{where } e_j = \delta_j\ (A_j - b_j) \qquad \text{for } j = 1,2 \ldots m \tag{5.4}$$

Term s_i is introduced since in the solution of non linear programming problems, setting of pots s may be different for different $\frac{\partial Z}{\partial x_i}$. It is clear that to instrument a non-linear programming problem we must be able to compute $\frac{\partial Z}{\partial x_i}$ & $\frac{\partial A_j}{\partial x_i}$ and hence the first derivative of Z must be

defined throughout the solution space and the first derivative of A_j must be defined in the neighbourhood of the boundary space (defined by strict equalities in 5.2).

5.3 Possible types of solution:

.1. The most straight forward solution occurs when the absolute maximum or minimum value of the objective function Z lies within the region specified by the constraints on $x_1, x_2 \dots x_n$. In this case the constraints do not play any significant part in the solution. This is only possible for a nonlinear objective function as the very nature of this type of solution implies the existence of stationary points in a domain. For this type of non-linear programming problem, the conditions $\frac{\partial Z}{\partial x_i} = 0$ are satisfied and integration ceases precisely when the x_i give the maximum/minimum value of the objective function.

.2. When the absolute maximum/minimum value of objective function Z lies outside the restricted region, the solution is entirely determined by the constraints $A_j \leq b_j$ for $j = 1,2 \dots m$.

The x_i's will be driven in the direction of maximum/minimum value of $Z(x)$ until a constraint boundary is encountered. Further motion is along the boundaries of the restricted region until the objective function is maximized/minimized. In practice an exact solution to the problem is never obtained for the following reasons:

(a) $\frac{\partial Z}{\partial x_i}$ can never be made zero since global maxima/minima lies outside the region defined by constraints.

(b) To make the input to the integrators as zero (for steady state solution), an error voltage is required to cancel $\frac{\partial Z}{\partial x_i}$ i.e. one or more of the constraints being minutely violated.

Due to the characteristics of diode feed back circuits providing an amplified error voltage Ke_j this violation in constraints is very small and the overall error does not exceed 1-2%.

5.4 Difficulties associated with non-linear programming:

As pointed out in chapter 2, solving nonlinear programming problems on the analog computer present two main difficulties.

(a) Non-linearities when described on analog computer require more elaborate equipment in the form of multipliers, Deviders and function generators etc. which restrict the size of nonlinear programming problems that can be solved on Analog Computer.

(b) Non linear programming problem may have a number of local maximum (Minimum) values which will warrant sufficient scanning of the region of interest to arrive at the absolute maximum/minimum. This is illustrated by a simple two dimensional example.

5.5 Example 6:*

$$\text{Maximize} \quad Z = 25\,(x_1-2)^2 + (x_2-2)^2 \tag{5.5}$$

*Hadley, G., "Nonlinear & Dynamic Programming" pp 11-12.

Subject to

$$
\begin{aligned}
A_1 &= -x_1 + x_2 \leq 2 \\
A_2 &= .5x_1 + .5x_2 \leq 3 \\
A_3 &= .333x_1 - x_2 \leq .667 \\
A_4 &= x_1 + x_2 \geq 2 \\
&\& \; x_1, x_2 \geq 0
\end{aligned}
\tag{5.6}
$$

The geometrical representation is presented in Fig.5.1. It may be seen that the extreme point 4 yields the global maximum of the objective function. However extreme point 2 yields a relative maximum of the objective function different from the global maximum.

5.5.1 Scaling:

Writing $y_i = \dfrac{x_i}{p_i}$ in (5.6)

$$
\begin{aligned}
-p_1y_1 + p_2y_2 &\leq 2 \\
.5p_1y_1 + .5p_2y_2 &\leq 3 \\
.333p_1y_1 - p_2y_2 &\leq .667 \\
p_1y_1 + p_2y_2 &\geq 2
\end{aligned}
\tag{5.7}
$$

Dividing each constraint of (5.7) by 5 and taking $p_1 = p_1 = 5$, the scaled problem is recorded in table 5.1. Expressing the objective function in terms of y_i

$$Z = 25\ (5y_1-2)^2 \div (5y_2-2)^2$$

$$\text{or} \quad \frac{Z}{1250} = Z' = .02\ (5y_1-2)^2 + .0008(5y_2-2)^2 \tag{5.8}$$

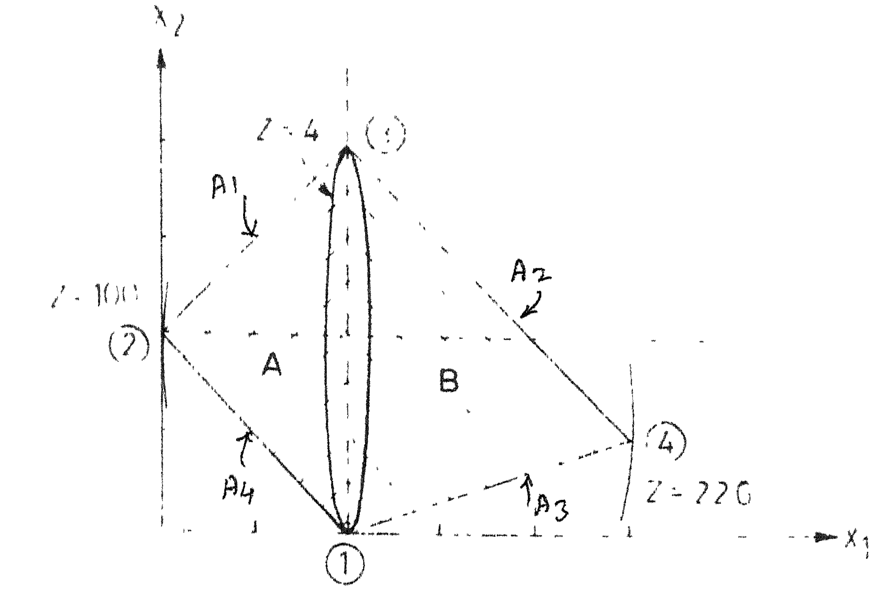

Fig.5.1 Geometrical representation of example 6

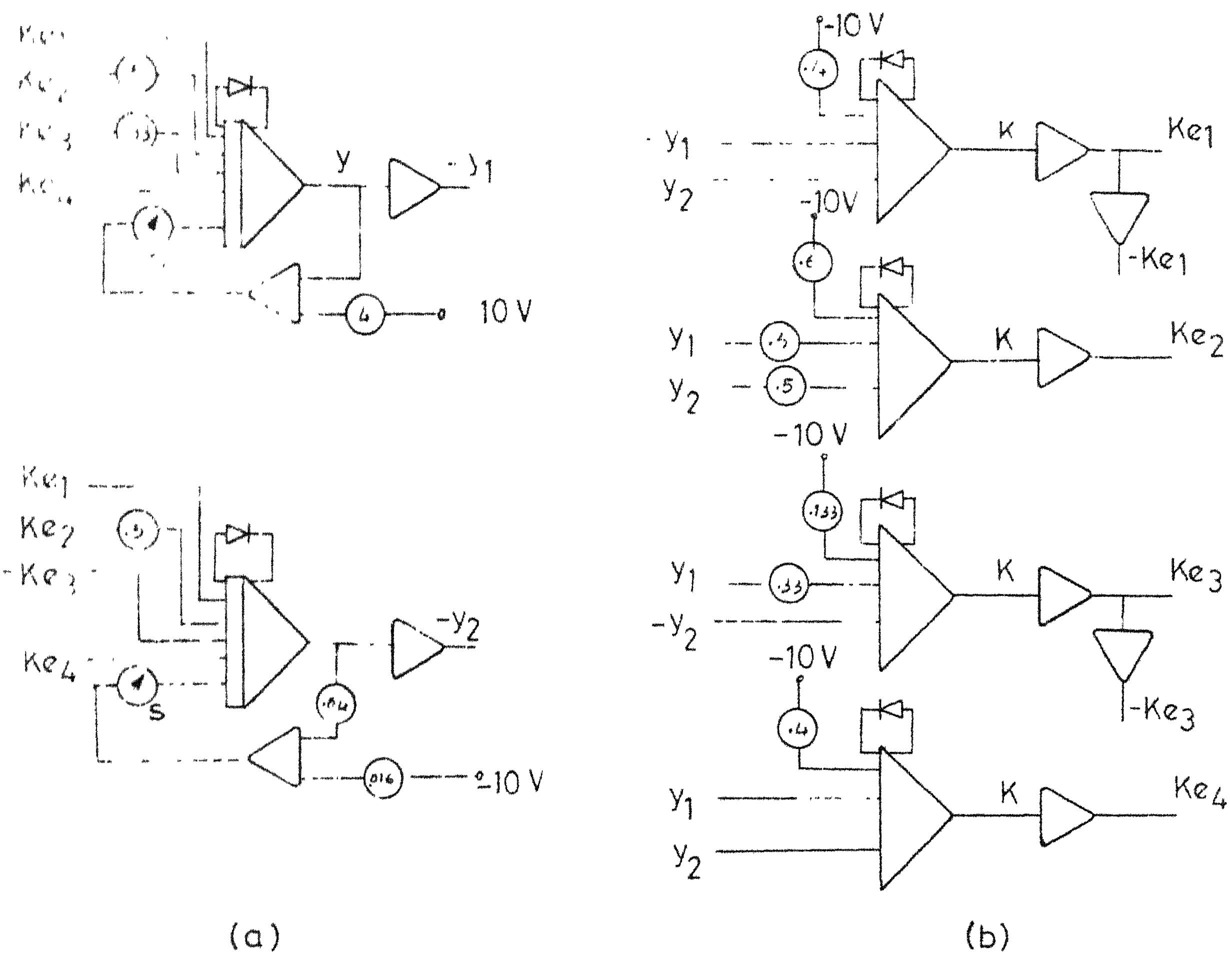

Fig. 5.2 Analog schematic for example 6.

Constraint No.	Variables		Specification
	y_1	y_2	
1	-1	1	$\leq .4$
2	.5	.5	$\leq .6$
3	.333	-1	$\leq .1334$
4	1	1	$\geq .4$

Table 5.1: Scaled constraint coefficients for example 6.

The partial derivatives of Z' with respect to y_1 & y_2 are

$$\frac{\partial Z'}{\partial y_1} = y_1 - .4$$
$$\frac{\partial Z'}{\partial y_2} = .04\, y_2 - .016 \qquad (5.9)$$

5.5.2 Analog Solution:

The scaled problem in table 5.1 and equations (5.8) and (5.9) are instrumented on the analog computer as shown in Fig.5.2. To obtain the optimum solution of nonlinear programming problems, procedure outlined in step 5 of 3.1.3 is slightly modified while setting the various s pots. Unlike in LP problem solution, where there is only one s pot, solution of nonlinear programming problem on the analog computer may involve n number of s_i pots to be set individually. The pots s_i may be set in the following steps.

1. Initially keep the setting of all s_i pots as unity and put the analog computer from "Reset" to "operate".

2. Vary s_1, at the same time observe $\frac{\partial Z}{\partial y_1}$. The desired setting of pot s_1 is the one that gives minimum value of $\frac{\partial Z}{\partial y_1}$. However in certain cases setting of the pot. s_i may affect other $\frac{\partial Z}{\partial y_i}$. In that case all such $\frac{\partial Z}{\partial y_i}$ should also be observed for a minimum value.

3. Step 2 is repeated for setting of other s_i $(i = 2,3 \ldots n)$ pots.

Thus several computer runs were made using modified procedure and with different sets of initial conditions. It was observed that whenever the starting point was in region A (fig. 5.1), the local maxima point 2 was obtained but with the starting point in region B, the global maxima point 1 was arrived at. The solution obtained with starting point in region A is

$$x_1 = .401 \times 5 = 2.005$$

$$x_2 = 0$$

whereas the solution obtained with starting point in Region B is

$$x_1 = .995 \times 5 = 4.975$$

$$x_2 = .201 \times 5 = 1.005$$

which is the global maxima for this problem.

5.6 Convexity and concavity:

It is at times worthwhile to test for the convexity/concavity of the objective function and of the constraints of the non-linear

programming problem. For the cases detailed below the local optimum point is also the global optimum point and hence no scanning of solution space is necessary.

(a) A maximization problem where the objective function is concave and the constraints are concave/linear.

(b) In a minimization problem when the objective function is convex and the constraints are convex/linear. In appendix 'E' the necessary and sufficient conditions for convexity/concavity are explained and a general scheme for assessing the convexity/concavity of functions of more than one variables is given with example . Three typical non linear programming problems that were solved on the TR-20 will be discussed now.

5.7 Example 7* (quadratic programming)

Quadratic programming problems are special class of non-linear programming problems in which the constraints are all linear but the objective function is of the quadratic form.

$$\text{Maximize} \quad Z = (12.1-x_1)x_1 + (8.4-2x_2)x_2 + (11.2-2x_3)x_3 + (7.7-x_4)x_4 + (10.4-x_5)x_5 \tag{5.8}$$

Subject to:

$$\begin{aligned} 7.1x_1 + 6.2x_2 + 5.2x_3 + 6.5x_4 + 7.4x_5 &\leq 20 \\ 5.7x_1 + 6.3x_2 + 6.8x_3 + 5x_4 + 6.3x_5 &\leq 20 \\ 2.2x_1 + 2.5x_2 + 3x_3 + 3.5x_4 + 1.3x_5 &\leq 25 \end{aligned} \tag{5.9}$$

---

* Vajda, "Readings in Mathematical programming" pp 117

5.7.1 Scaling:

Dividing (5.9) by 10 and writing $y_i = \frac{x_i}{2.5}$

the scaled constraints are tabulated in table 5.2

Constraint No.	Variables					Specification
	y_1	y_2	y_3	y_4	y_5	
1	.71	.62	.52	.65	.74	$\leq$ .8
2	.57	.63	.68	.5	.63	$\leq$ .8
3	.22	.25	.3	.35	.13	$\leq$ 1

Table 5.2: Scaled constraints coefficients for example 7

Writing Z from (5.8) in terms of y_i and dividing Z by 50

we get

$$\frac{Z}{50} = Z' = (.605-.125y_1)y_1 + (.42-.25y_2)y_2 + (.56-.25y_3)y_3$$
$$+(.385-.125y_4)y_4 + (.52-.125y_5)y_5 \tag{5.10}$$

Differenciating Z'

$$\frac{\partial Z'}{\partial y_1} = .605 - .25y_1$$
$$\frac{\partial Z'}{\partial y_2} = .42-.25y_2$$
$$\frac{\partial Z'}{\partial y_3} = .56-.25y_3 \tag{5.11}$$
$$\frac{\partial Z'}{\partial y_4} = .385-.25y_4$$
$$\frac{\partial Z'}{\partial y_5} = .52-.25y_5$$

5.7.2 Analog Solution:

The Analog Computer diagram to instrument the scaled problem represented by table 5.2 and equations (5.10) and (5.11) is shown in Fig. 5.3. Modified procedure as outlined in 5.5.2 was followed and the results obtained are tabulated in table 5.3. Digital computer results are also tabulated alongwith for comparison.

Variables	Scaled solution	Analog solution	Digital solution	Error
x_1	.593	1.4825	1.47	.85%
x_2	.043	.1075	.10	7.5%
x_3	.456	1.14	1.13	.88%
x_4	0	0	0	-
x_5	.165	.412	.42	1.9%

Table 5.3: Comparison of Analog & Digital solutions for example 7

5.7.3 Effect of Varying RHS elements:

The effect of variation of RHS elements on the objective function was studied at discrete points. One element of RHS vector was varied at a time with the help of calibrated potentiometer and the corresponding objective function values recorded in table 5.4. The values of the variables were observed simultaneously and any changes from the basic feasible solution recorded in the remarks column of table 5.4. The variations are plotted in Fig.5.4

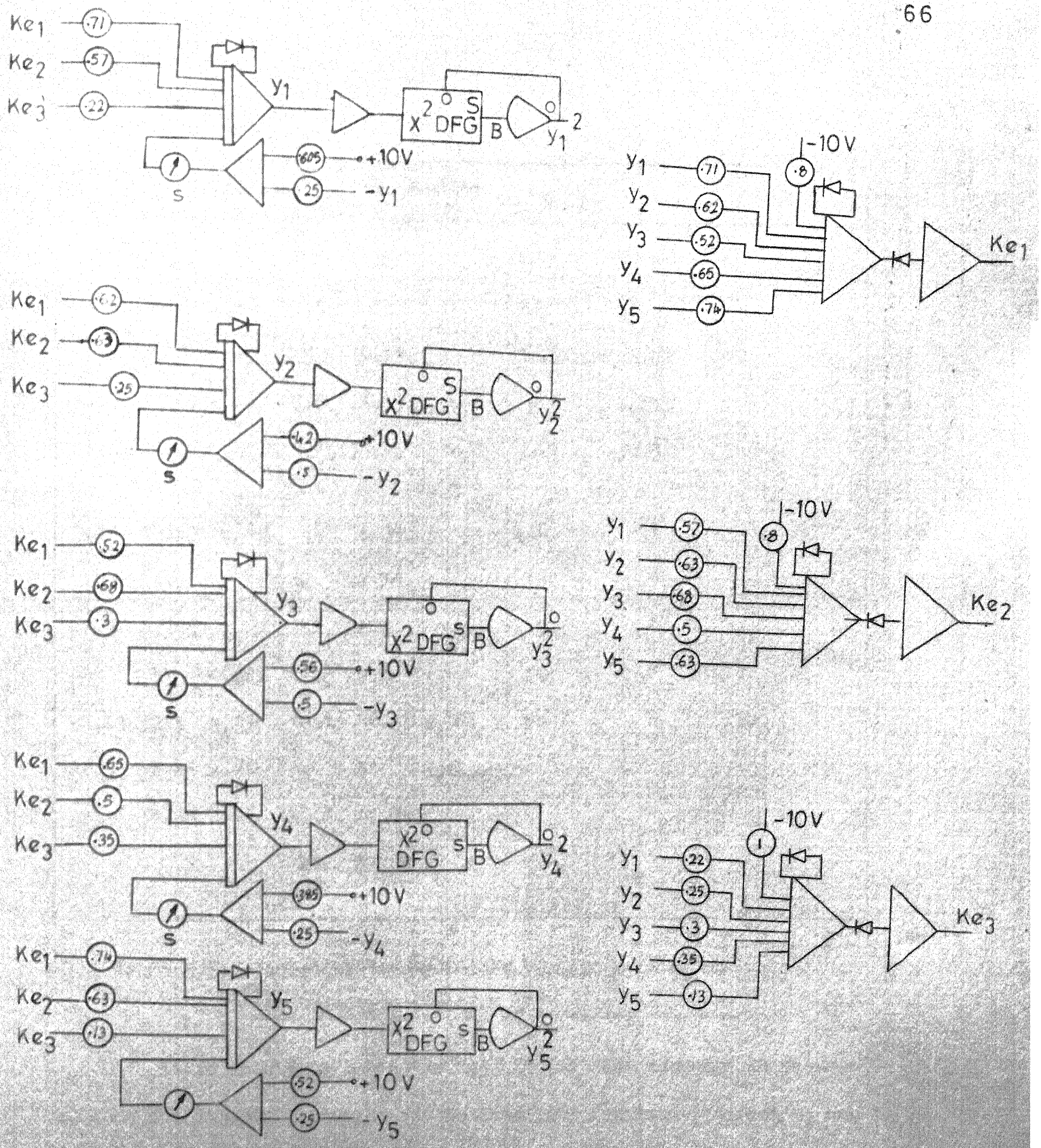

Fig.5.3 Analog schematic for example 7

b_1	Z	Remarks	b_2	Z	Remarks	b_3	Z	Remarks
0	0		0	1.35		1.25	16.25	
3.75	7.83	$x_1=0$	2.5	6.25		2.5	22.85	
5	9.83		5	10.75		3.75	26.53	
7.5	13.67		7.5	14.93	$x_4 = 0$	5	29.13	
10	17.4		8.75	16.84	$x_3 = 0$ $x_5 = 0$	5.87	30.3	
11.87	19.85	$x_2=0$	10	18.85		7.5	31	
15	24.3		12.5	22.78		10	"	
17.5	27.65	$x_5=0$	15	26.5		15	"	
20	31	Opt.	17.5	29.8	$x_2=0$	20	"	
21.25	32.4	$x_4>0$	18.13	30.33	$x_4>0$	25	"	Opt.
22.5	33.23	$x_2>0$	20	31	Opt.			
25	33.88		22.5	31.95				
			25	31.95				

Table 5.4: Record of the effect of variation in b_j on the objective function (Z)

It is seen from table 5.4 and Fig.5.4 that increase in value of b_3 does not improve the value of the objective function. Infact b_3 can be reduced from 25 to 7.5 without affecting the value of Z. A further increase in b_2 from 20 results in a nominal increase in the value of Z to 31.95

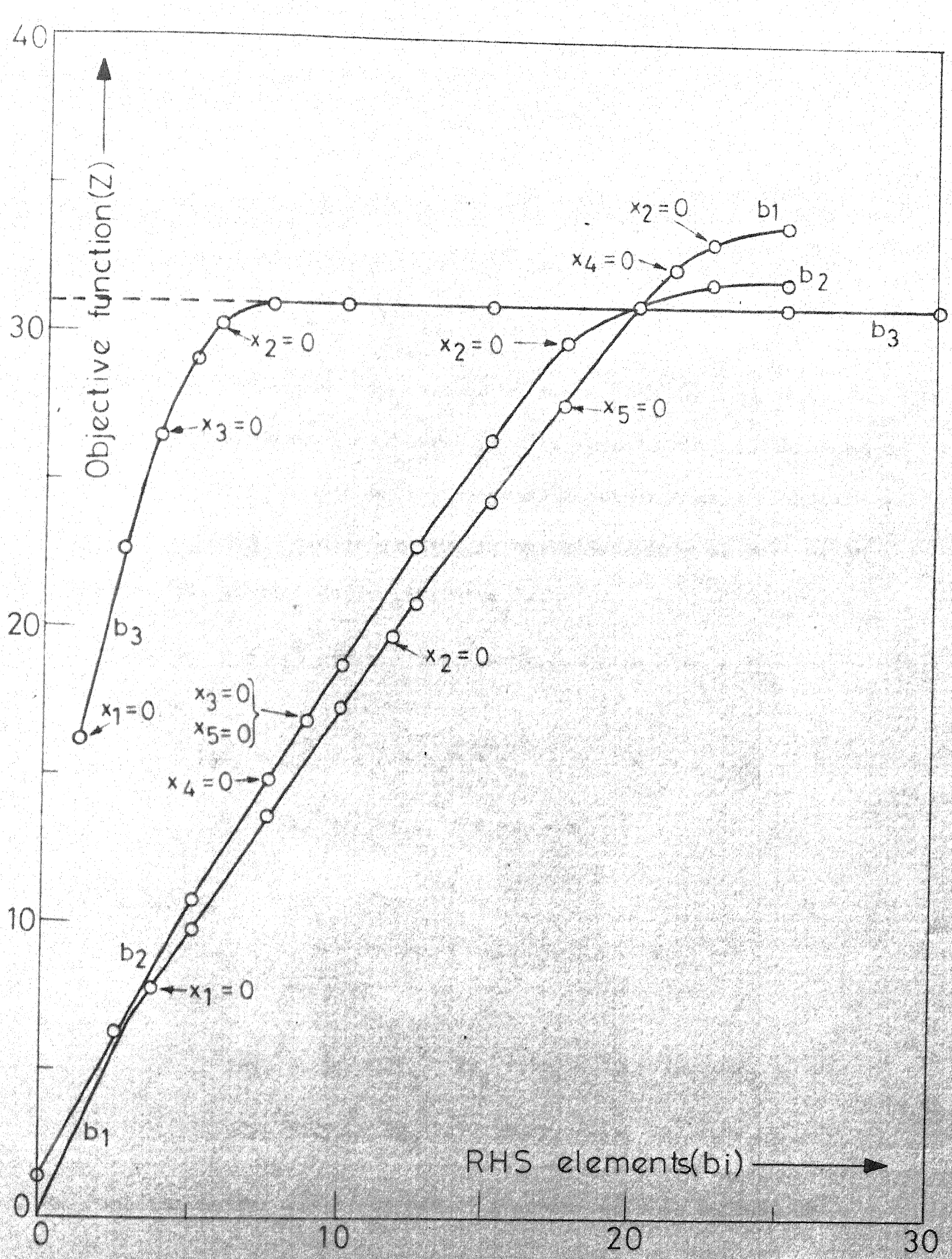

Fig.5.4 Plots of objective function(Z) vs. RHS elements(bi).

and is constant thereafter with further increase in b_2. However increase in b_1 from 20 does result in increase of the objective function. It can be pointed out that any reduction in the value of b_1 & b_2 from 20, results in reduction in the value of the objective function.

5.8 Example 8*:

We now consider an example in which both the objective function and the constraints are nonlinear. It will be seen that patching up of non-linear constraints on Analog computer requires a more elaborate set up to cater for the nonlinearities in the constraints as well as in computing the partial derivative of A_j's.

$$\text{Minimize} \quad Z = x_1^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4 \tag{5.12}$$

subject to

$$\begin{aligned} A_1 &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 \leq 8 \\ A_2 &= 2x_1^2 + x_2^2 + x_3^2 + 2x_1 - x_2 - x_4 \leq 5 \end{aligned} \tag{5.13}$$

5.8.1 Scaling:

Writing $y_i = \frac{x_i}{2}$ in (5.13) and dividing it by 20

we get

$$\begin{aligned} .2y_1^2 + .2y_2^2 + .2y_3^2 + .2y_4^2 + .1y_1 - .1y_2 + .1y_3 - .1y_4 &\leq .4 \\ .4y_1^2 + .2y_2^2 + .2y_3^2 + .2y_1 - .1y_2 - .1y_4 &\leq .25 \end{aligned} \tag{5.14}$$

The Jacobian matrix $J(A,x)$ of set of function $A(x)$ is defined by

* "Operation Research" Vol.17.2, 1969. pp, 978

The Jacobian matrix J (A,x) of set of function A(x) is defined by

$$J(A,x) = \begin{bmatrix} \frac{\partial A_1}{\partial x_1} & \frac{\partial A_1}{\partial x_2} & \cdots & \frac{\partial A_1}{\partial x_n} \\ \frac{\partial A_2}{\partial x_1} & \frac{\partial A_2}{\partial x_2} & \cdots & \frac{\partial A_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial A_m}{\partial x_1} & \frac{\partial A_m}{\partial x_2} & \cdots & \frac{\partial A_m}{\partial x_n} \end{bmatrix}$$

and is required to be computed in the feed back circuits of programming problems having nonlinear constraints. For this problem from (5.14)

$$J(A,x) = \begin{bmatrix} .4x_1+1 & .4x_2-.1 & .4x_3+1 & .4x_4-.1 \\ .8x_1+.2 & .4x_2-.1 & .4x_3 & -.1 \end{bmatrix} \tag{5.15}$$

Writing Z from (5.12) in terms of y_i and scaling it, we get

$$\frac{Z}{50} = Z' = .08y_1^2+.16y_2^2+.08y_4^2-.2y_1-.2y_2-.84y_3+.28y_4 \tag{5.16}$$

Differentiating Z'

$$\begin{aligned} \frac{\partial Z'}{\partial y_1} &= .16y_1-.2 \\ \frac{\partial Z'}{\partial y_2} &= -.2 \\ \frac{\partial Z'}{\partial y_3} &= .32y_3 -.84 \\ \frac{\partial Z'}{\partial y_4} &= .16y_4 + .28 \end{aligned} \tag{5.17}$$

5.8.2 Analog Solution: The Analog computer diagram instrumenting the scaled problem represented by relations (5.14), (5.15) & (5.17) is shown in Fig.5.5. This problem utilized 40 amplifiers with seven multipliers, four squarers and 38 potentiometers. The optimum point was obtained by using modified procedure outlined in 5.5.2 and the results are tabulated in table 5.5 alongwith the digital computer results.

Variables	Scaled Analog solution	Analog solution	Digital solution	Error
x_1	0	0	0	-
x_2	.515	1.03	1	3%
x_3	1.00	2.00	2	Nil
x_4	-.494	-.988	-1	1.2%
Z	.	-44.8	-44	1.81 %

Table 5.5: Comparison of Analog & digital solutions for example 8

5.8.3 Effect of Varying RHS Elements: The effect of variation of RHS vector on the objective function was studied at discrete points with the help of calibrated potentiometers and the corresponding values of the objective function recorded in table 5.6. Remarks column of table 5.6 indicate any changes that occur in the basic feasible solution. These variations are plotted in Fig.5.6. In Fig.5.6, negative of the objective function is plotted against b_1 and b_2, because the optimum value of Z is in the negative range. It can be seen that the value of the objective function

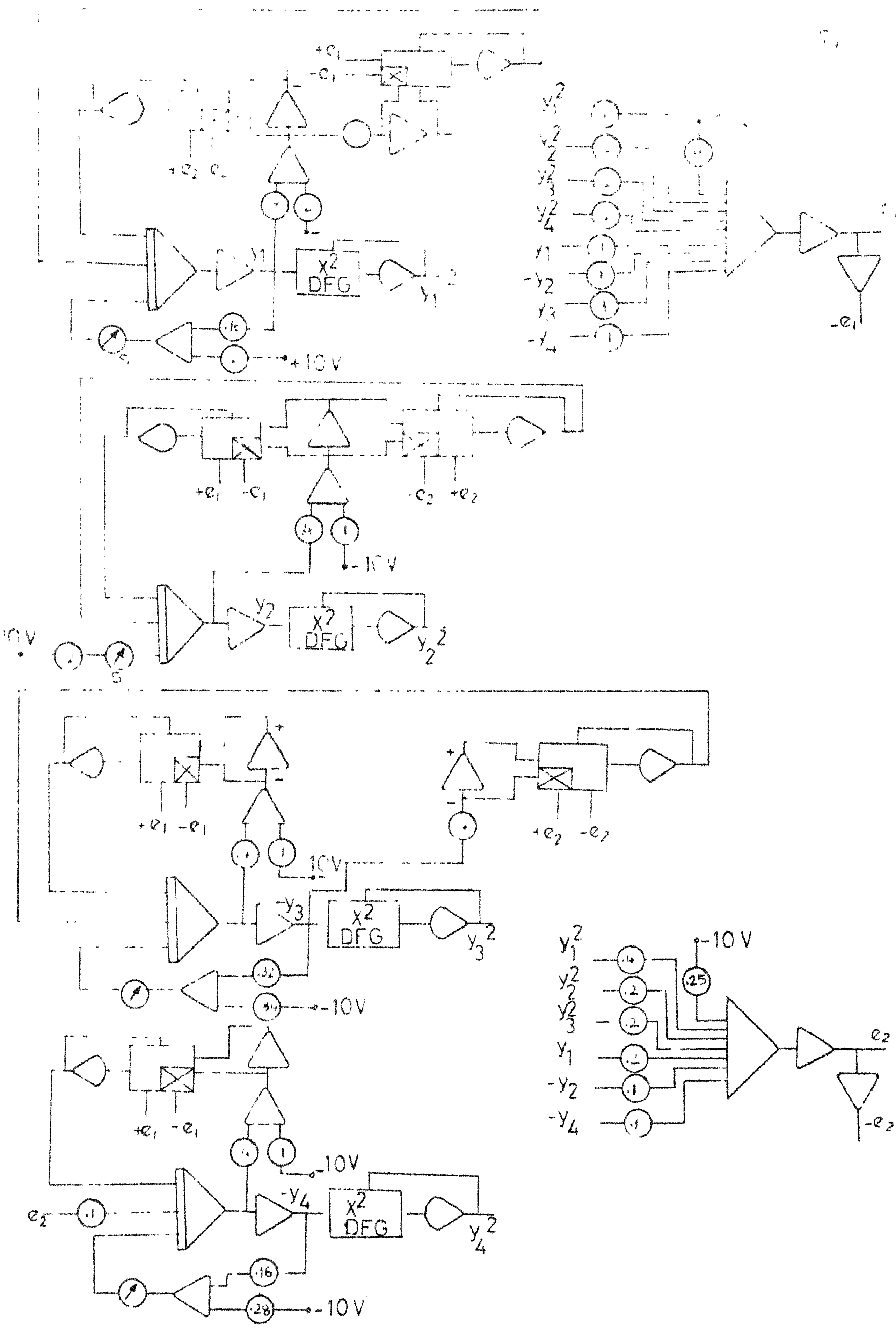

Fig. 5.5 Analog computer set-up for example 8.

b_1	Z	Remarks	b_2	Z	Remarks
2	-10.175		0	-29.7	
3.8	-33.85	$x_1 > 0$	2	-35.9	
4	-44.25	$x_1 > 0$	4	-42.5	
6	-44.75		5	-44.8	Optimal
8	-44.8	Optimal	6	-47.1	
10	-44.75		7	-49.25	
12	-44.7		7.3	-49.9	
14	-44.6		8	-50.5	$x_1 > 0$
16	-44.5				

Table 5.6: Record of the effect of the variation in b_j (j=1,2) on the objective function (Z)

can be significantly reduced by further increasing the value of b_2 from 5, whereas an increase/decrease in b_1 from 8 increases the value of the objective function.

5.9 Optimum Generator Allocation: This problem forms part of load flow studies of power networks. The total amount of real power in the network emanates from the generators, the location and size of which are fixed. The generation must equal the demand plus the losses at each instant and this generated power must be divided between the generators in a unique ratio in order to achieve most economic operation. The optimality criteria may be in terms of cost or in terms of transmission losses, both of which

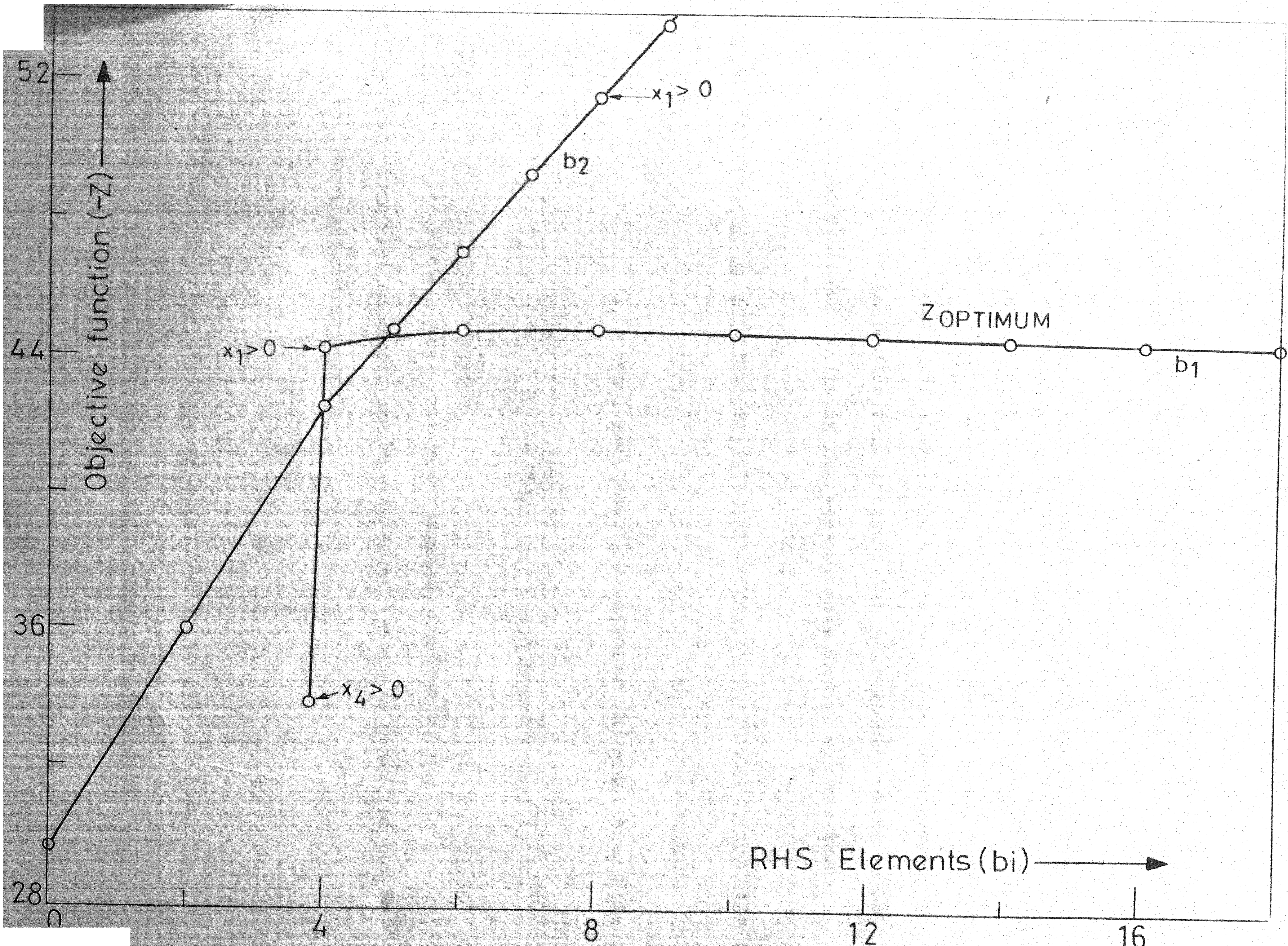

Fig.5.6 Plots of objective function (Z) vs. RHS elements (bi)

can be expressed with fair accuracy in terms of real power generation. It is always necessary to keep the voltage levels of certain buses at close tolerances, which is achieved by proper scheduling of reactive powers. The reactive generation in general does not have significant influence on the cost or transmission losses because they are controlled by varying the field excitation. Thus excluding the reactive powers and associated voltage profile from our nonlinear programming problem we can state the nonlinear constraint that must be satisfied at all the times

$$\sum_{i=1}^{n} P_i - P_D - P_L = 0 \tag{5.18}$$

where

$\sum_{i=1}^{n} P_i$ = Total real power generation at 'n' generating stations

$P_D = \sum_{i=1}^{m} P_{Di}$ = Total real power demand from m loads.

P_L = Total real power Transmission losses.

These transmission losses can be expressed in terms of generated powers by Transmission-loss formula

i.e $P_L = \sum_i \sum_j P_i B_{ij} P_j$

where

P_i, P_j = source loadings

B_{ij} = Transmission-loss formula coeffieients

5.10 Example 9* We consider a 5 bus problem as shown in Fig.5.7. The reactive power and associated voltage profile is fixed for each bus. The real power demand at each bus is indicated in Fig.5.7. It is required to find generations P_1, P_2 & P_3 in order to minimize transmission losses. The loss coefficients for the system are assumed to be known.

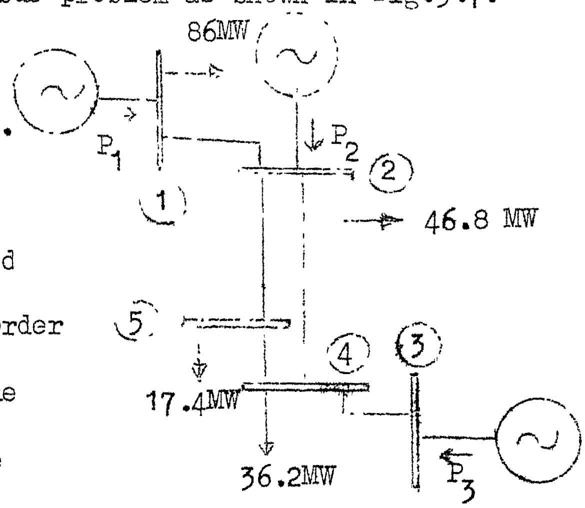

Fig.5.7: Real power demand at 5 busses

The transmission losses expressed in terms of loss co-efficient formula are:

$$P_L = P_1^2 B_{11} + P_2^2 B_{22} + P_3^2 B_{33} + 2P_1P_2{+}B_{12} + 2P_2P_3 B_{23} + 2P_3P_1B_{31} \tag{5.19}$$

$$\begin{aligned} \text{where} \quad B_{11} &= .02725 \\ B_{22} &= .0309 \\ B_{33} &= .323 \\ B_{21} = B_{12} &= -.0035 \\ B_{31} = B_{13} &= -.0360 \\ B_{32} = B_{23} &= -.00565 \end{aligned} \tag{5.20}$$

All powers are expressed as p.u. taking 200 MW as base power.

*Kirchmayer, L.K, Economic operation of power system, pp.118.

5.10.1 Analog Solution: The problem can be expressed as:-

$$\text{Minimize} \quad Z = .02725P_1^2 + .0309P_2^2 + .323P_3^2 - .007P_1P_2 - .0113P_2P_3 - .0736P_3P_1 \tag{5.21}$$

subject to the constraint

$$A = P_1+P_2+P_3 - (.43+.234+.181+.087) - .02725P_1^2 - .0309P_2^2 - .323P_3^2 + .007\,P_1P_2+.0113P_2P_3+.0736P_3P_1 = 0 \tag{5.22}$$

We write

$$\begin{aligned} \frac{\partial Z}{\partial P_1} &= .0545P_1 - .007P_2 - .0736P_3 \\ \frac{\partial Z}{\partial P_2} &= .0618P_2 - .007P_1 - .0113P_3 \\ \frac{\partial Z}{\partial P_3} &= .646P_3 - .0113P_2 - .0736P_1 \end{aligned} \tag{5.23}$$

Also

$$\begin{aligned} \frac{\partial A}{\partial P_1} &= 1-.0545\,P_1 + .007P_2 + .0736P_3 \\ \frac{\partial A}{\partial P_2} &= 1-.0618\,P_2 + .007\,P_1 + .0113\,P_3 \\ \frac{\partial A}{\partial P_3} &= 1-.646\,P_3 + .0113\,P_2 + .0736P_1 \end{aligned} \tag{5.24}$$

The Analog computer set up for this problem is shown in Fig.5.8. This problem utilised 26 amplifiers with 6 multipliers 3 squarers and 32 pots. The optimum solution was obtained by using modified procedure outlined in 5.5.2. The result obtained are tabulated in table 5.7 along with the digital computer results.

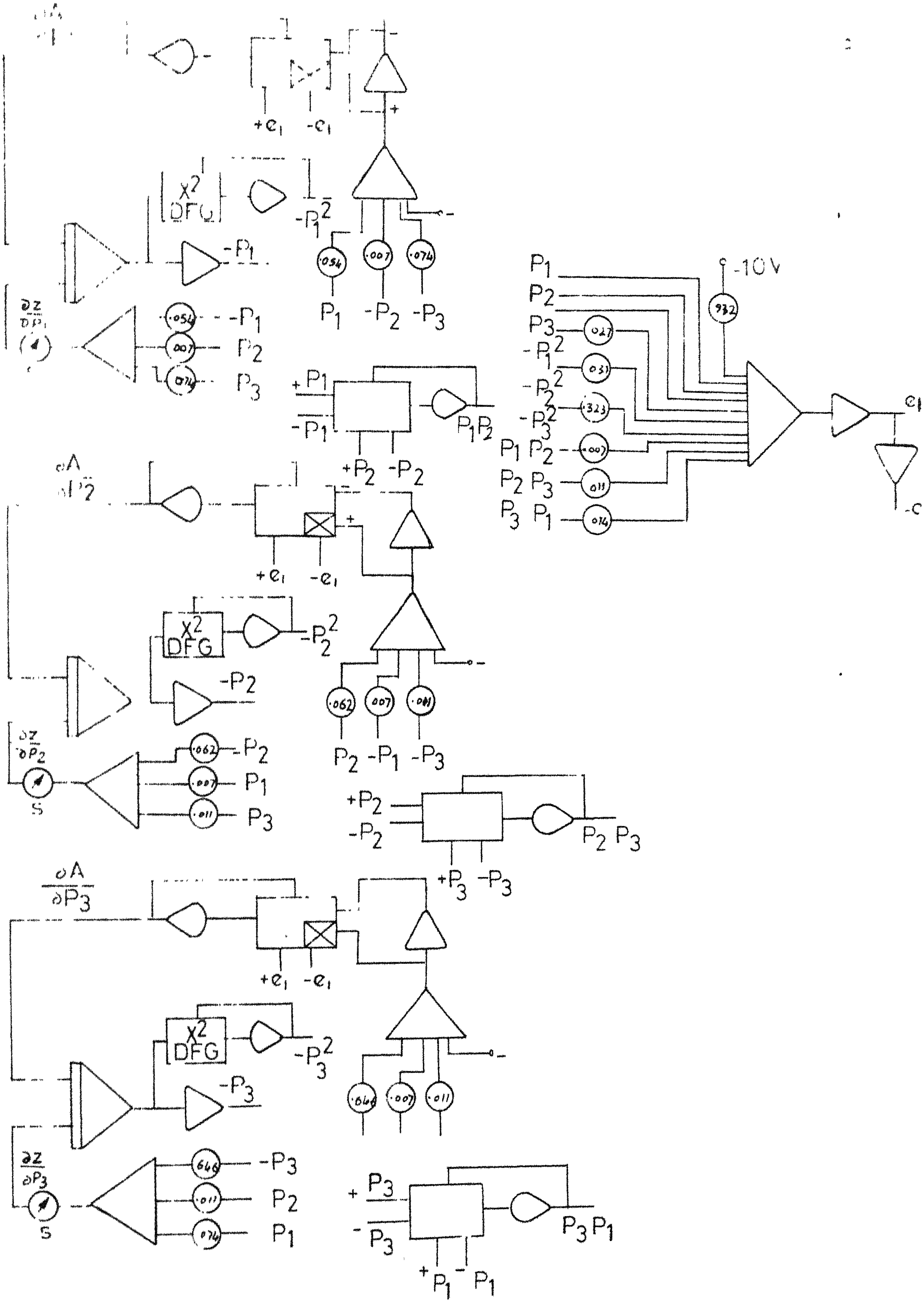

Fig.5.8 Analog schematic for example 9.

Variables	Scaled Analog solutinn	Analog solution	Digital solution
P_1	.67	134 MW	144 MW
P_2	.12	24 MW	10.668 MW
P_3	.157	31.4 MW	34.6 MW

Table 5.7: Comparison of Analog & Digital solutions for example 9.

Total optimum generated power $= P_1+P_2+P_3 = 189.4$ MW

Total Demand .. $= 186.4$ MW

Hence Total Losses $P_L \simeq 189.4-186.4 = 3$ MW

Whereas, as calculated from objective function (5.21), $P_L = 2.62$ MW. This discrepancy in P_L is attributed to the small 'error voltage' that occur in the solution of Analog Computer. Since the total demand P_D varies with time, this aspect can be taken care of by varying the appropriate pot. setting and observing the new optimum solution. Thus plot can be made of P_i's for different values of loads.

5.11 Discussion: It is clear from the examples that nonlinear programming problems which are expressible with the available analog computer components, can be solved on analog computer with ease. The size of non linear programming problem that can be solved on analog computer is restricted by the available nonlinear components e.g. multipliers, dividers and function generators etc. Some of the nonlinear programming

problems may need sufficient scanning of the solution space to arrive at the global optimum. In certain cases the test for convexity-concavity does eliminate the scanning process. A logrithmic function generator was assembled from the data given in "Maintenance Manual for TR-20". The details are given in appendix 'F.'

CHAPTER - VI

6.1 The Analog computer method described in the previous chapter can be used in solving a variety of optimization problems that occur in many industries. The use of this analog method with slight modifications can also be extended in solving

(a) Simultaneous algebraic equations

(b) Matrix inversion problems

6.2 Simultaneous Algebraic Equations:

A set of simultaneous algebraic equations can be written as,

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots\cdots a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots\cdots a_{2n} x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1} x_1 + a_{n2} x_2 + \cdots\cdots a_{nn} x_n &= b_n
\end{aligned}
\tag{6.1}
$$

which may be expressed in matrix form as

$$AX = B \tag{6.2}$$

In order that (6.2) have a solution, the determinant $|A|$ must be non-vanishing

For solution of (6.1) on the analog computer, the set of equations is first transformed into a set of differential equations as:

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots\cdots a_{1n} x_n - b_1 &= -\dot{x}_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots\cdots a_{2n} x_n - b_2 &= -\dot{x}_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1} x_1 + a_{n2} x_2 + \cdots\cdots a_{nn} x_n - b_n &= -\dot{x}_n
\end{aligned}
\tag{6.3}
$$

The LHS of each equation of (6.3) represent the input to an integrator having one of the unknowns $x_1, x_2 \ldots\ldots x_n$ as its output. The use of integrators would ensure that all residual errors due to the method of analog computer set up would be zero. This reduces to the original set of equations when all derivatives are zero. The matrix operation being performed is

$$\left[A + sI \right] X = B \tag{6.4}$$

The Analog Computer is being used to solve a set of n simultaneous differential equations. The steady state solution of which (if it exists) will be the solution of original set of algebraic equations. The Analog Computer set up is then a dynamic system described by (6.4), the stability of which may be determined by finding the roots of the characteristic equation.

$$\det \left[A + sI \right] = 0 \tag{6.4a}$$

The dynamic system described by (6.4) will be stable only if all the roots $s_1, s_2 \ldots s_n$ of (6.4a) have negative real parts i.e. if matrix A is positive definite.

6.2.1 Analog set up:

If A is a nonsingular square matrix of order n, then $C = A^T A$ is always positive definite* . Therefore the concept of premultiplying by the transpose of A always lead to a stable computer system, if a solution exists. Therefore expression (6.2) becomes

---

* Hausner, A. "Analog & Analog/hybrid computer programming, Theorem 11.8 pp. 322.

$$A^T A X = A^T B \tag{6.5}$$

This set is then transformed as in expression (6.3) into a set of simultaneous differential equations

$$A^T A X - A^T B = -\dot{X} \tag{6.6}$$

The computer set up corresponding to expression (6.6) is always stable since all roots of the characteristic equation $\left| A^T A + sI \right| = 0$ are real negative (the matrix $A^T A$ is positive definite). Using analog computer one can proceed in two different ways.

(a) Calculate the new matrix $A^T A$ and the column vector $A^T B$ and set up the set (6.6) on the computer. This will require only n integrator for the set up. For sets having a large number of unknowns, a considerable amount of calculation has to be carried out before starting the proper solution on the computer.

(b) The multiplication of A^T can be performed on the Analog computer by solving two sets of equations

$$AX - B = e \tag{6.7a}$$

$$A^T e = -\dot{X} \tag{6.7b}$$

which are equivalent to the original set (6.6). A general set up diagram for expressions (6.7a) & (6.7b) is shown in Fig.6.1(a) and (b) respectively. In order that the error voltages e_i are very nearly reduced to zero, they are multiplied by large constant K (usually 10 or 100) before being fed back to the integrators. The Analog Computer set of Fig.6.1 differs from Fig.2.1 in the following respect.

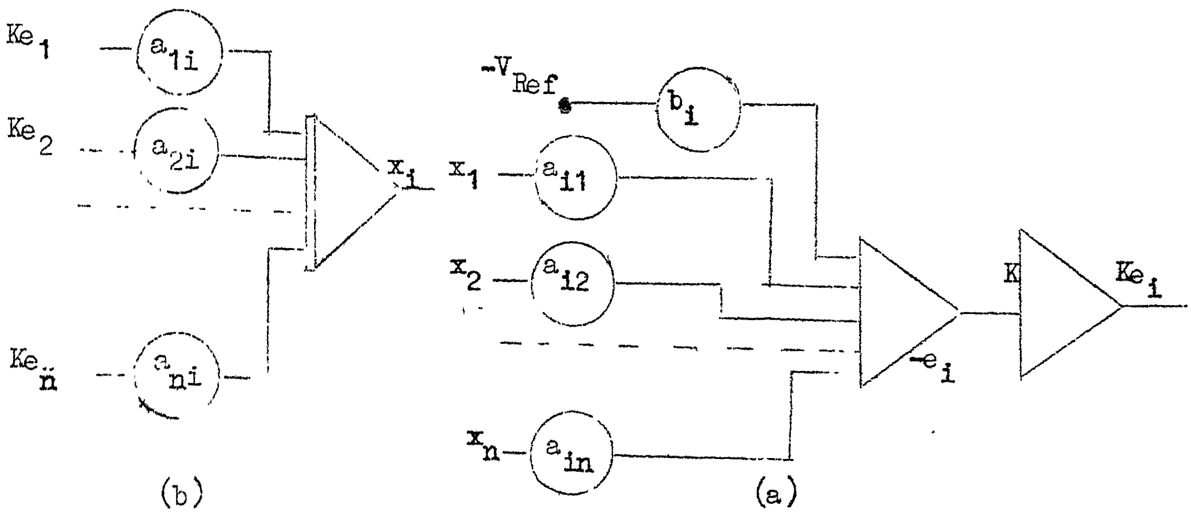

Fig.6.1: Partial diagram for general method of solving simultaneous algebraic equations

(a) Diodes are not used in Fig.6.1

(b) Partial derivative of objective function$\left(\frac{\partial Z}{\partial x_i}\right)$ is omitted from Fig.6.1 for obvious reasons.

6.2.2 Improving Accuracy:

If more accurate results are desired than obtainable with Analog Computer set up of Fig.6.1, an iterative procedure may be used to improve accuracy. Let the values of x_i's obtained by computer be represented by column matrix X' and let the error be represented by column matrix E. Then the correct result is

$$X = X' + E$$

and the matrix operation desired is

$$A X = B$$

$$\text{i.e.} \quad A\left[X' + E\right] = B$$

$$\text{or} \quad AE = B - AX' \qquad (6.8)$$

Thus with the values of X' obtained from the computer set up, the new column matrix $\left[B - AX'\right]$ can be calculated and the values of these elements substituted for the RHS coefficients B in the same computer set up. The corrected values of column vector X are then obtained by adding the newly computed value of error vector E. A simple example will illustrate the analog method.

6.2.3 Example 10:

Solve

$$\begin{aligned}
x_1 + 10\,x_2 + x_3 &= 10 \\
2x_1 + 20\,x_3 + x_4 &= 10 \\
3x_2 + 30\,x_5 + 3x_6 &= 0 \\
10x_1 + x_2 - x_6 &= 5 \\
2x_4 - 2x_5 + 20x_6 &= 5 \\
x_3 + 10x_4 - x_5 &= 0
\end{aligned} \qquad (6.9)$$

To scale it, each equation of 6.9 is divided by such a constant so as to make all the coefficients less than or equal to unity. These scaled equations from (6.9) are written in the matrix form as follows:

$$\begin{bmatrix} .1 & 1 & .1 & 0 & 0 & 0 \\ .1 & 0 & 1 & .1 & 0 & 0 \\ 0 & .1 & 0 & 0 & 1 & .1 \\ 1 & .1 & 0 & 0 & 0 & -.1 \\ 0 & 0 & 0 & .1 & -.1 & 1 \\ 0 & 0 & .1 & 1 & -.1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ .5 \\ 0 \\ .5 \\ .25 \\ 0 \end{bmatrix} \tag{6.10}$$

The scaled computer diagram corresponding to (6.10) is shown in Fig.(6.2). With the first solution vector X' for K = 100, vector $[B - AX']$ was calculated and the error vector obtained in the second run. The corrected results are tabulated in table 6.1. The digital computer result are also tabulated for comparison sake.

Variables	Analog solution corrected by one iteration		Digital solution
x_1	.432 + .0005	= .4325	.43335
x_2	.911-.0005	= .9105	.91071
x_3	.461-.002	= .459	.45954
x_4	-.05 -.006	= -.056	-.05750
x_5	-.101 -.012	= -.113	-.11549
x_6	.2495-.0065	= .243	.24420

Table 6.1: Comparison of analog & digital solution for example 10

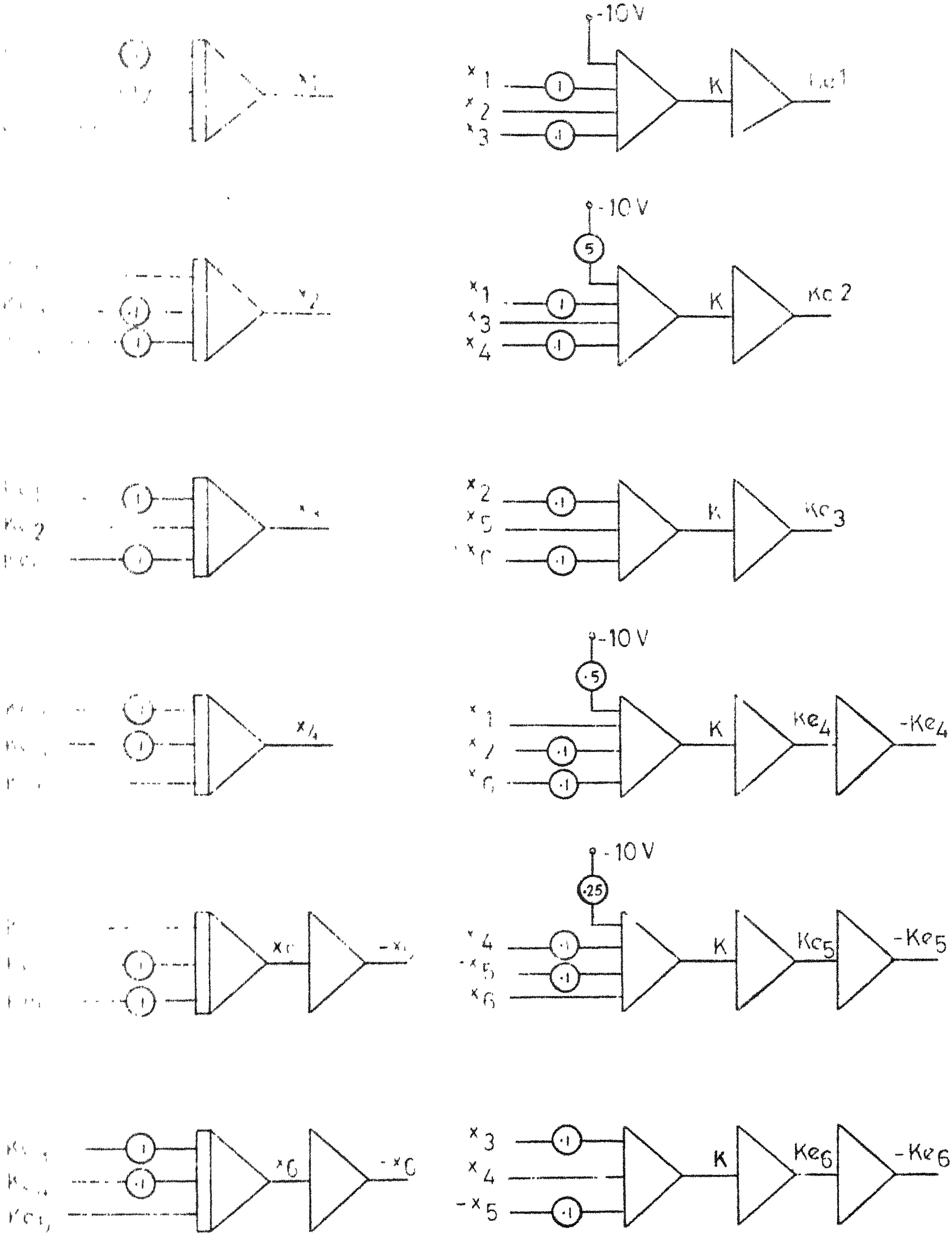

Fig. 6.2 Analog computer set-up for example 10

6.3 Matrix Inversion:

Matrix inversion on Analog Computer can be accomplished in the same way as the set of simultaneous algebraic equations are solved. Elements of A^{-1} are computed from

$$A\,A^{-1} = I \tag{6.11}$$

Let $A^{-1} = K$ then 6.11 becomes

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

which can be written as nxn linear algebraic equations. The first n equations are

$$\begin{aligned} a_{11}\,k_{11} + a_{12}\,k_{21} + \cdots\cdots\; a_{1n}k_{n1} &= 1 \\ a_{21}\,k_{11} + a_{22}\,k_{21} + \cdots\cdots\; a_{2n}k_{n1} &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n1}\,k_{11} + a_{n2}\,k_{21} + \cdots\cdots\; a_{nn}\,k_{n1} &= 0 \end{aligned} \tag{6.12}$$

Set (6.12) when solved on analog computer as described in section 6.2 will solve for the first column of matrix K. Other columns of matrix K can easily be computed, one at a time, from the same analog computer set up of (6.12) by proper substitution of unity elements on RHS. Thus in n runs inverse of the matrix A can be computed.

6.4 Discussion:

Analog solution of simultaneous algebraic equations is advantageous in cases where there are several sets to be solved with the same matrix. The Analog Computer components required for this set up are

(a) Amplifiers = $3n + m$

(b) Potentiometer = $2g + h$

where m = number of rows plus columns containing negative coefficients

n = number of variables in the problem

g = Non zero elements of matrix A.

h = Non zero elements of column vector B.

With proper scaling, a large number of elements of matrix A can be made unity thereby reducing the number of potentiometers required for the set up, which otherwise limit the size of the problem set that can be solved on analog computer. The use of external equipment for additional inputs and additional coefficients potentiometers designed in Chapter II and proper scaling can easily result in the solution of problem with 10 variables or so.

CHAPTER VII

CONCLUSION:

A simple, fast and reasonably accurate method of solving medium sized Linear & Non-linear Programming problems using the Analog computers TR-20 has been described in detail alongwith a number of illustrations. The main limitation of this general purpose Analog computer in solving programming problems, is the limited number of available potentiometers and the availability of restricted number of inputs to each amplifier. This has been partially overcome by the use of an external equipment, designed from the indigenous components to allow variable banks of sixty additional inputs of gain 1 and gain 10, alongwith sixty additional potentiometers. The time required in setting these external potentiometers can be cut down by the use of "single press, double throw" switches. This press button switch can be connected in such a way that, in its normal condition, the input is connected to the external pot, which gets disconnecte on pressing the switch and instead -10V is applied to the potentiometer to facilitate the pot. setting.

An important aspect in obtaining the analog computer solution, is the scaling procedure, which has also been described for linear and non-linear programming problems.

For linear programming problems and some of the non-linear programming problems, the Analog solution is quickly obtained in a single run, in a matter of few seconds. However, sufficient scanning of the

solution space to find global optima is nescesitated for a large number of non-linear programming problems, which too can be carried out in a relatively short time.

In the Analog Computer method described, the effects of variation of model parameters on the objective function are readily observed by adjusting the relevant coefficient potentiometer. Thus one can very conveniently explore the sensitivity of the solution as a function of system parameters and/or observe the effects of changes predicted for future on the system itself. This gives a very valuable insight into the practical programming problems and here the Analog Computer definetely has an edge over the digital computer.

Analog computers have been pushed into the background with the advent of digital computers. However, digital computers, at times, are not easily available and readily accessible. Moreover the cost associated with the use of digital computer is quite exhorbitant. Analog computer on the other hand, is not so expensive and its flexibility combined with ease of programming provide theoretical-cum-experimental approach to gain a real insight into the behaviour of the system under study. Therefore with limited accuracy as its limitation, it can prove to be very useful in obtaining a first order solution to be refined later on.

A further study of other optimization techniques given out in (4)* and (6)* using Analog/hybrid computer would motivate the design of more equipment to further enhance the capabilities of available TR-20.

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*Bibliographical references.

BIBLIOGRAPHY

1. Albow & G. Brigham, "An Analog solution of programming problems" operation Research Vol.3, Nov. 1955, pp.338.

2. Borsky V. & J. Matyas, "Computation by Electronic Analogue Computers" Iliffe Books Ltd. 1968.

3. Firth, AWO, "Optimization Problems: Solution by an Analog Computer" computer Journal, Vol.4, 1961, pp. 68.

4. Hausner, A. "Analog & Analog/hybrid programming", Prentice-Hall Inc. 1971.

5. Jackson, A.S., "Analog Computation", Mc-Graw Hill Book Company Inc. 1960

6. Korn, G.A. & T.M. Korn, "Electronic Analog & Hybrid computers", Mc-Graw Hill Book Company Inc. 1972.

7. Longley, D, "An Analog Computer application in operation Research", Elect. Engg., Vol.36, June 19C4, pp. 378-381.

8. Mc-Millon, C. Jr., "Mathematical programming", John-Wiley & Sons, 1970.

9. Pyne, I.B., "Linear programming on an Electronic Analog Computer", AIEE Transactions (Part I) Vol.75, 1956, pp.139.

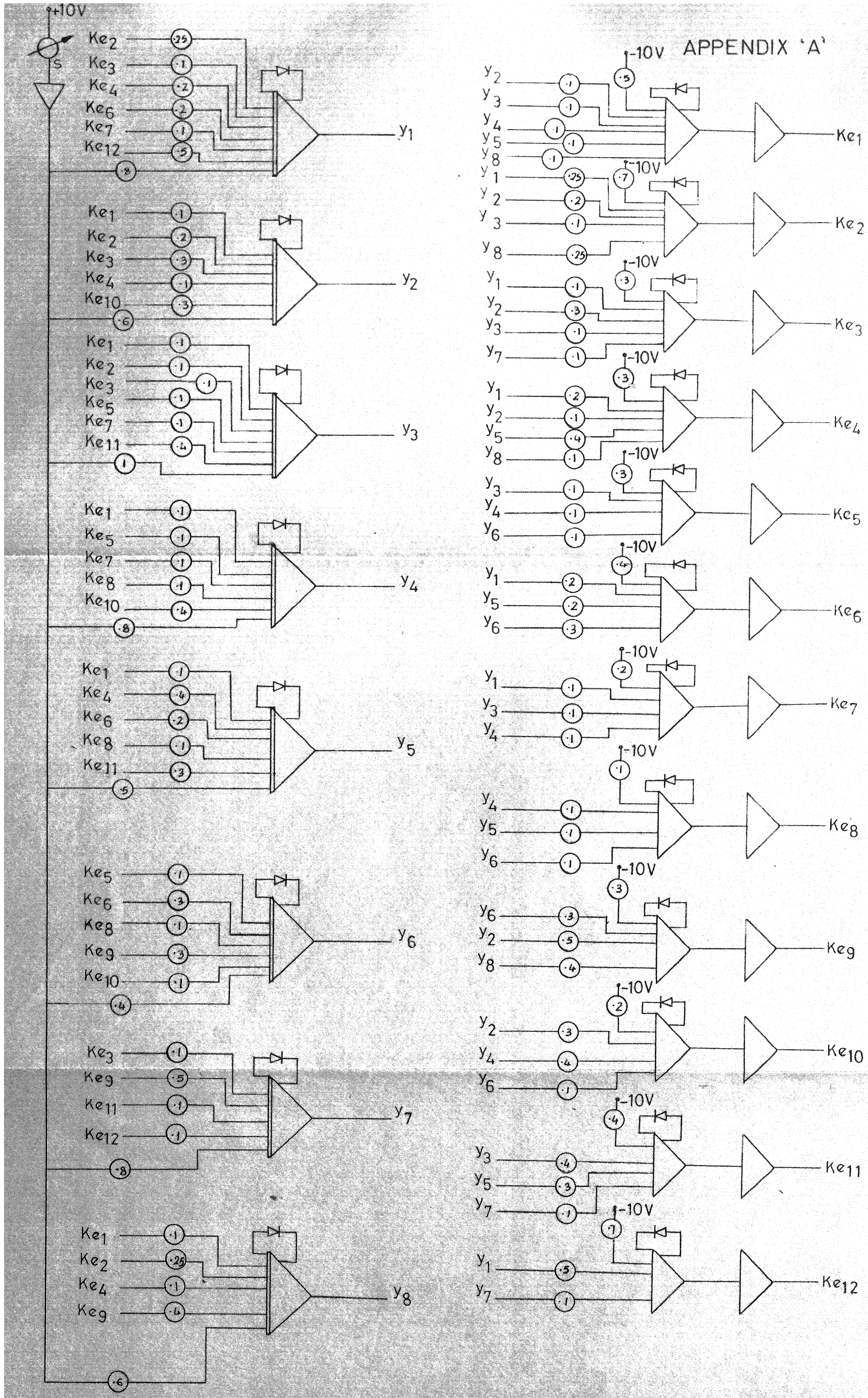
Other related papers:

1. Eilon S & D.P. Deziel, "The use of Analog Computer in some OR problems", Simulation, Sept. 1967 pp.121.

2. Neustadt, L.W., "Application of Linear & Nonlinear Programming Techniques", Proceedings 3rd International Analog Computation meeting Sept. 1961.

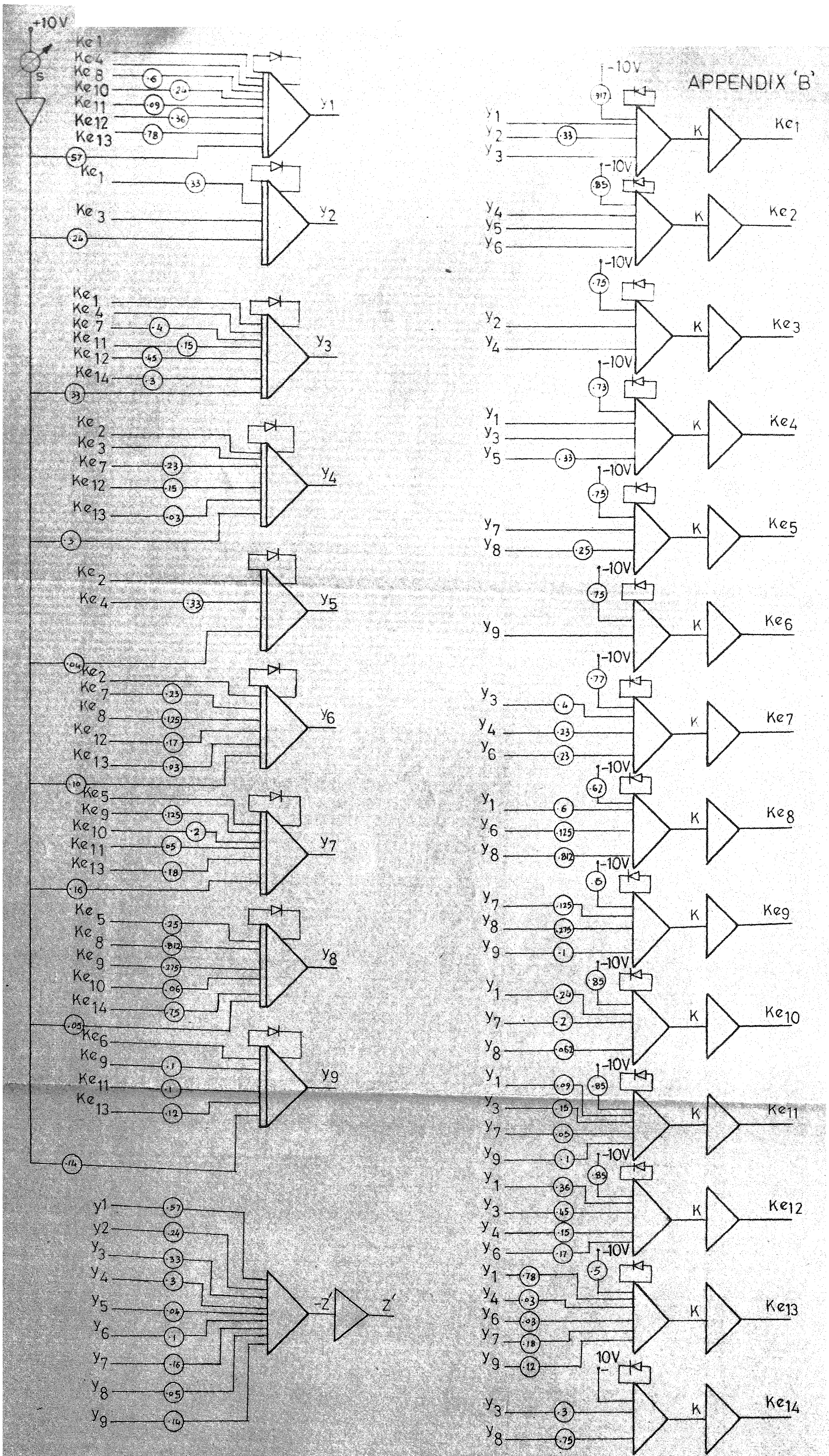
3. Moser, J.H. etall, "Non linear programming technique for Analog Computation", Chem. Engg. Prog., Vol.57, Jan. 1961.

APPENDIX 'A'



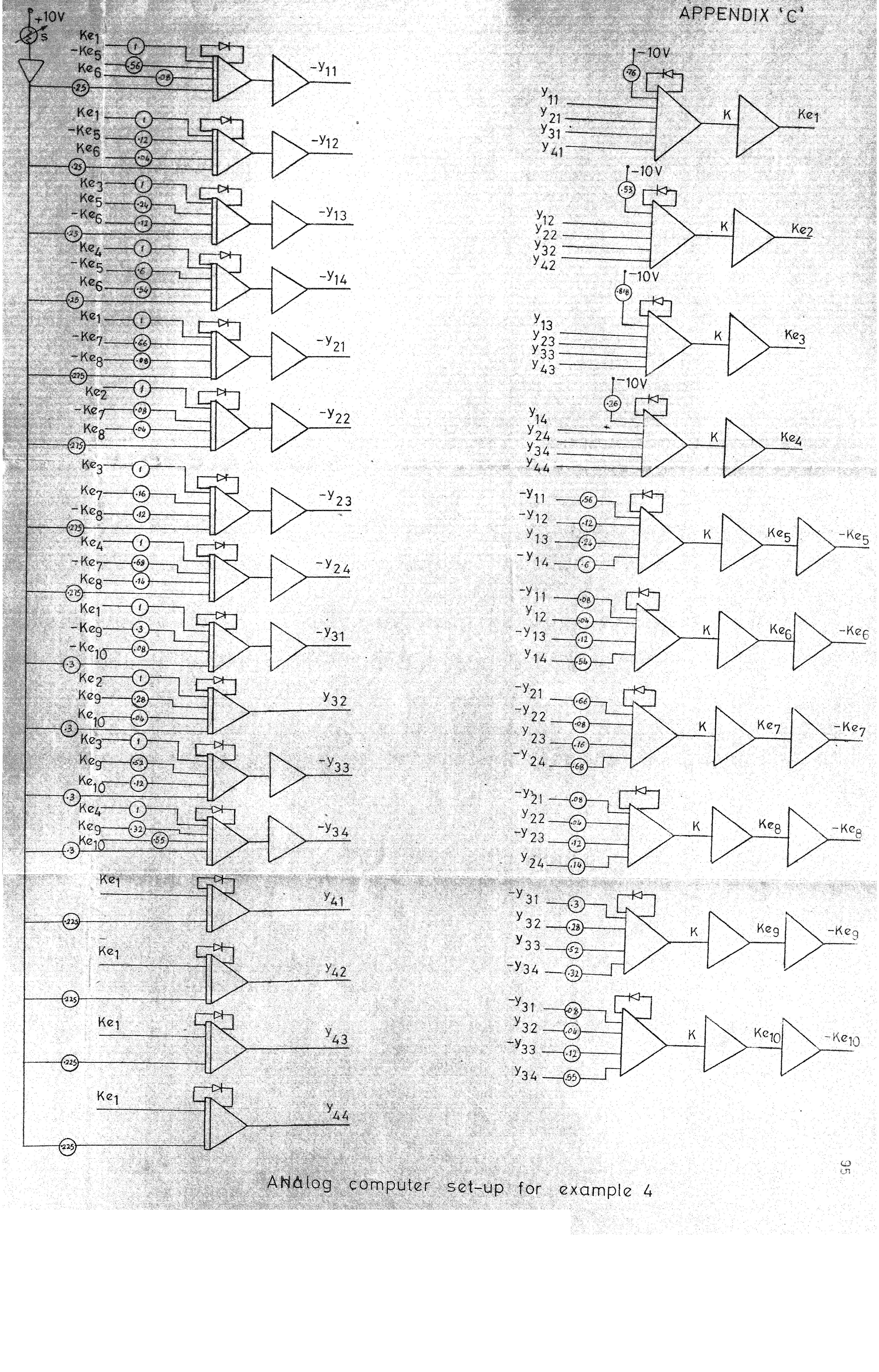
Analog set-up for example 2.

APPENDIX 'B'



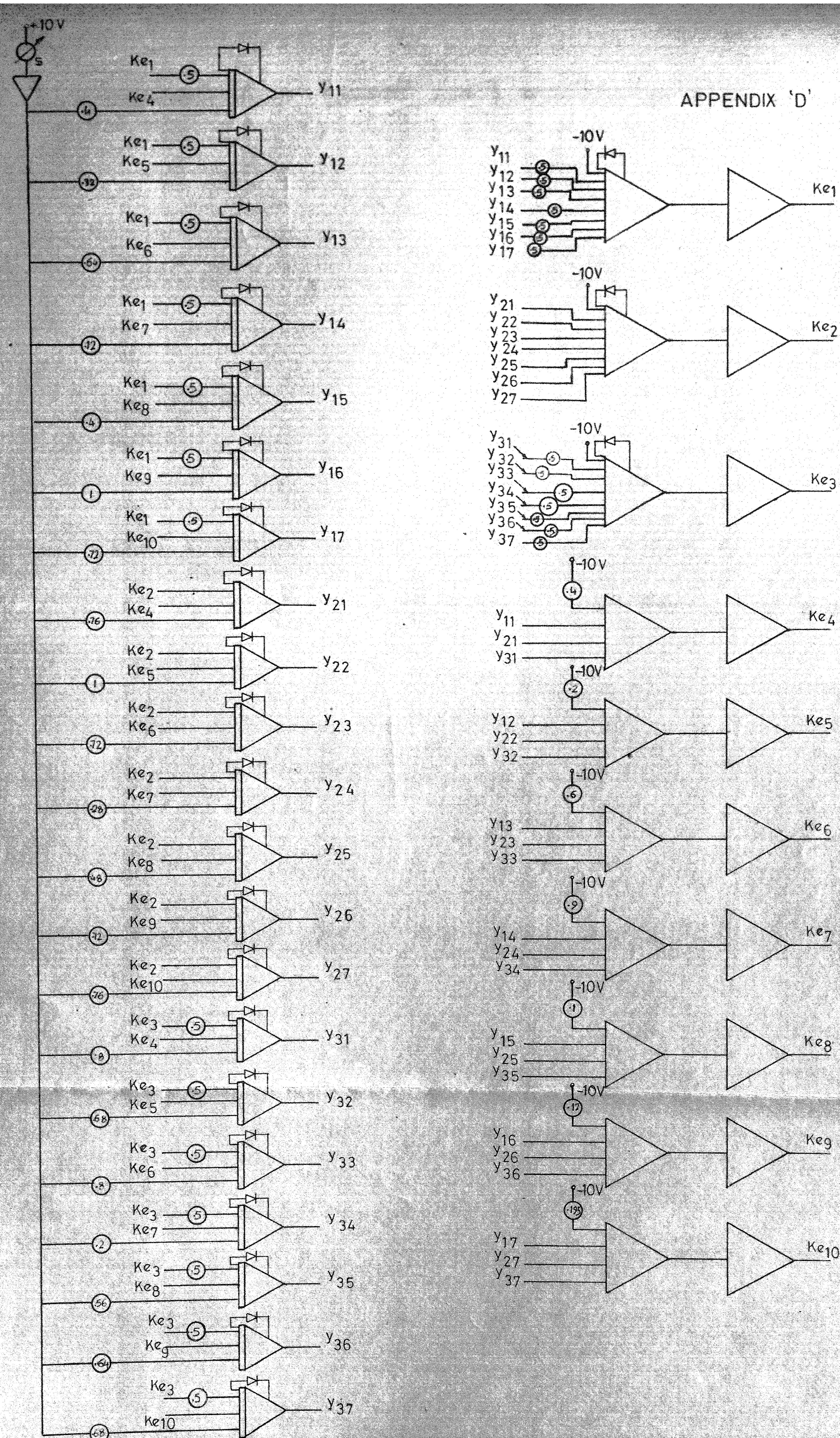
Analog computer set-up for example 3

APPENDIX 'C'



ANAlog computer set-up for example 4

APPENDIX 'D'



Analog computer set-up for example 5

APPENDIX 'E'

TESTING FOR CONVEXITY-CONCAVITY

A function of one variable $f(X)$ is a convex function if for each pair of values of X, say X' & X"

$$f\left[\lambda X'' + (1-\lambda)X'\right] \leq \lambda f(X'') + (1-\lambda) f(X') \qquad (1)$$

where $0 \leq \lambda \leq 1$

It is strictly convex if the strict inequality holds in (1) If $\leq$ in (1) is replaced by $\geq$, then $f(X)$ is a concave function. Again for the strict inequality the function is strictly concave. Fig.E.1 shows a convex function.

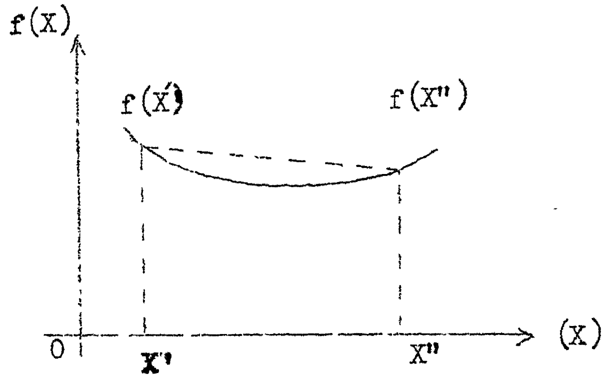


Fig.E.1: Convex function

$[X', f(X')]$ is a point on the curve and $[X'', f(X'')]$ is another point on the curve both of which are connected by straight line shown dashed. The expression $\left[\lambda X'' + (1-\lambda)X', \ \lambda f(X'') + (1-\lambda) f(X')\right]$ represents the points on the dashed line between points $[X', f(X')]$ and $[X'', f(X'')]$, $0 \leq \lambda \leq 1$. Thus all the line segment connecting the two points lie above the curve of $f(X)$. A convex function is always bending upward. A concave function always bends down word as shown in Fig.E.2.

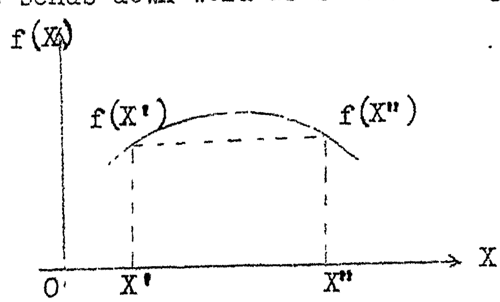


Fig.E.2: Concave Function

n variable function:

To assess convexity-concavity of a function of n variables $f(X_1, X_2 \ldots X_n)$, we define the determinant of the matrix of second order partial derivatives

$$\triangle = \begin{vmatrix} \frac{\partial^2 f}{\partial X_1^2} & \frac{\partial^2 f}{\partial X_1 \partial X_2} & \cdots\cdots & \frac{\partial^2 f}{\partial X_1 \partial X_n} \\ \frac{\partial^2 f}{\partial X_2 \partial X_1} & \frac{\partial^2 f}{\partial X_2^2} & \cdots\cdot & \frac{\partial^2 f}{\partial X_2 \partial X_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial X_n \partial X_1} & \frac{\partial^2 f}{\partial X_n \partial X_2} & \cdots\cdot & \frac{\partial^2 f}{\partial X_n^2} \end{vmatrix}$$

The principal minors of the matrix of second order partial derivative are then identified as:

$$\triangle_1 = \frac{\partial^2 f}{\partial X_1^2} \qquad \triangle_2 = \begin{vmatrix} \frac{\partial^2 f}{\partial X_1^2} & \frac{\partial^2 f}{\partial X_1 \partial X_2} \\ \frac{\partial^2 f}{\partial X_2 \partial X_1} & \frac{\partial^2 f}{\partial X_2^2} \end{vmatrix}$$

$$\triangle_3 = \begin{vmatrix} \frac{\partial^2 f}{\partial X_1^2} & \frac{\partial^2 f}{\partial X_1 \partial X_2} & \frac{\partial^2 f}{\partial X_1 \partial X_3} \\ \frac{\partial^2 f}{\partial X_2 \partial X_1} & \frac{\partial^2 f}{\partial X_2^2} & \frac{\partial^2 f}{\partial X_2 \partial X_3} \\ \frac{\partial^2 f}{\partial X_3 \partial X_1} & \frac{\partial^2 f}{\partial X_3 \partial X_2} & \frac{\partial^2 f}{\partial X_3^2} \end{vmatrix} \quad \cdots\cdots\cdots \triangle_n$$

The function in convex if $\Delta_1 > 0$, $\Delta_2 > 0$ . . .

The function is concave if $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$ . . .

(i.e. odd minors are < 0 & the even minors are > 0)

Example:

$$f = X_1^2 + 2X_2^2 + X_3^2 + X_1X_2 - 2X_3 - 7X_1 + 12$$

$$\frac{\partial f}{\partial X_1} = 2X_1 + X_2 - 7 \quad ; \quad \frac{\partial^2 f}{\partial X_1^2} = 2 \quad ;$$

$$\frac{\partial f}{\partial X_2} = 4X_2 + X_1 \quad ; \quad \frac{\partial^2 f}{\partial X_2^2} = 4 \quad ;$$

$$\frac{\partial f}{\partial X_3} = 2X_3 - 2 \quad ; \quad \frac{\partial^2 f}{\partial X_3^2} = 2 \quad ;$$

$$\frac{\partial^2 f}{\partial X_1 \partial X_2} = 1 \quad ; \quad \frac{\partial^2 f}{\partial X_1 \partial X_3} = 0 \quad ;$$

$$\frac{\partial^2 f}{\partial X_2 \partial X_3} = 0 \quad ; \quad \frac{\partial^2 f}{\partial X_2 \partial X_1} = 1 \quad ;$$

$$\frac{\partial^2 f}{\partial X_3 \partial X_2} = 0 \quad ; \quad \frac{\partial^2 f}{\partial X_3 \partial X_1} = 0$$

Hence $\Delta = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix}$

Thus $\Delta_1 = 2 > 0$; $\Delta_2 = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8-1 = 7 > 0$;

$\Delta_3 = 16 + 0 + 0 - (0+0+2) = 14 > 0$ and therefore

function f is convex

Assessing cnvexity-concavity for function of degree two is a simple matter regardless of number of variables. For functions of degree higher than two, the determinant of the matrix of second order partial derivatives, expanded by principal minors, may yield as many inequalities as there are variables. Under these conditions, specifying the domain of the variables for which convexity - concavity can be assumed becomes rather involved.

APPENDIX 'F'

ASSEMBLY OF A LOGARITHMIC DIODE FUNCTION GENERATOR

Principle:

Use of diodes as switching elements in straight-line-segment function generation has been found to be quite satisfactory. Any function of a monotonic nature can be generated by biased diode networks placed in a parallel configuration in the input network of an amplifier as shown in Fig.F.1.

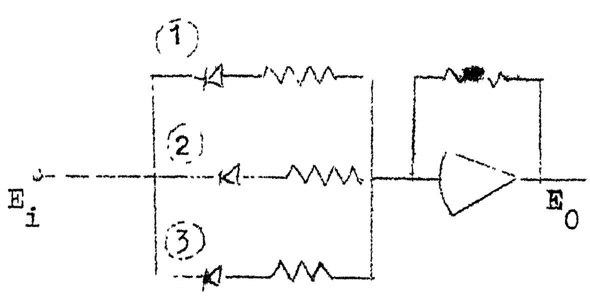


Fig.F.1: A Monotonic function generator

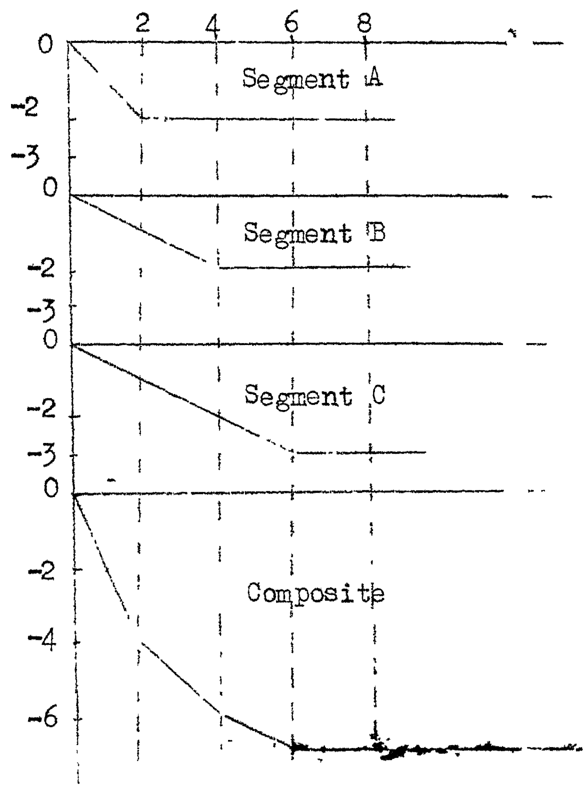


Fig.F.2: Plot of monotonic function

The resistor values of the network can be chosen so that the biased diode networks (1), (2) & (3) contribute the segments A, B & C as shown in Fig.F.2. When E_i is zero, all the three diodes are forward biased and

conducting. Therefore the input impedance to the amplifier is the parallel combination of the three branch impedances, giving the slope of segment 1 in Fig.F.2. As the input voltage E_i is made more and more positive, the diodes get cut off, one at a time. Everytime a diode gets cut off, the equivalent input impedance of the amplifier is increased, thereby decreasing the slope of the generated function. Thus Log X function, which has a decreasing slope as the input voltage X increases, can be generated from such a network.

Logarithmic function Generator:

The assembled LOG X Diode function generator utilizes a DC amplifier to produce an output voltage that is proportional to the logrithm of input signal vltage X. The output of this function generator is formed by seven straight-line voltage segments that closely approximate a logrithmic curve for a single polarity input voltage. Two indapendent logarithmic function generators were assembled. One generator accepts a positive input voltage X and produces a negative output voltage $Y = -5 \log_{10} 10 X$. The second generator accepts a negative input voltage X and produces a positive output voltage $Y = +5 \log_{10} 10X$. The circuit diagram of the two logarithmic function generator is shown in Fig.F.3*. Terminals S & 0 of Log X DFG are connected to terminals B & 0 respectively of a DC amplifier of TR-20 Analog computer. The feedback element for the amplifier is a stable 5K resistor. The remaining biased diode network constitutes the input impedance to the amplifier. For the positive input DFG, as the input voltage X is near zero, all diodes are conducting

*TR-20 Maintenance Manual, Drawing No.C016133 OS"

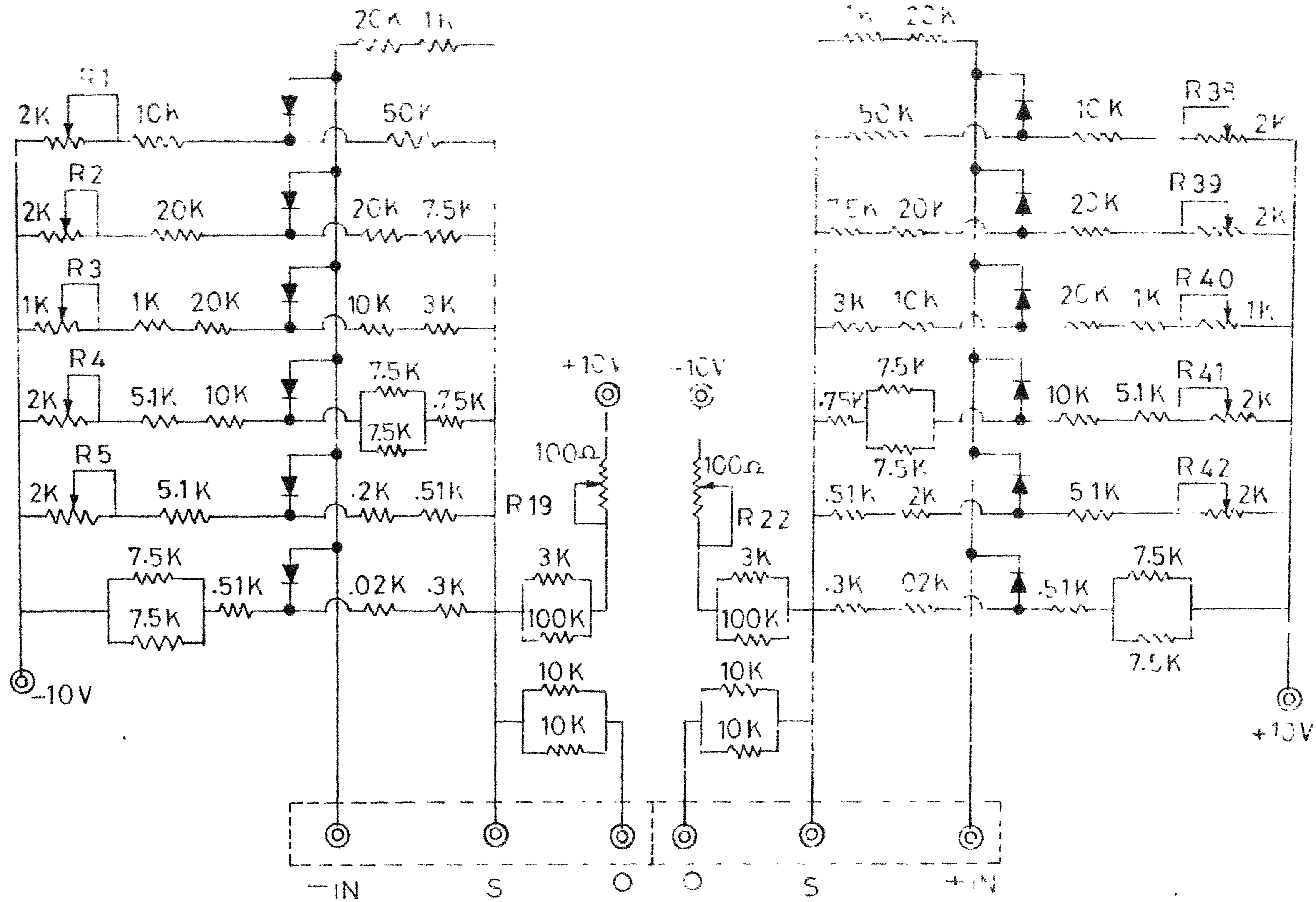


Fig. F- Schematic diagram for log X diode function generator.

2. Balance the amplifiers and check that the reference voltages of ± 10 volts are within specifications.

	Adj. No.	Input voltage V_{in}	Adjust	Output voltage V_o
Negative output DFG	1	+ 2.15	R_{19}	-.28 ± .01
	2	+ 5.23	R_5	-1.18 ±.01
	3	+ 6.92	R_4	-2.47 ±.01
	4	+ 8.13	R_3	-4.28 ±.01
	5	+ 9.05	R_2	-6.51 ±.01
	6	+ 9.79	R_1	-9.13 ± .01
Positive output DFG	1	-2.15	R_{22}	+.28±.01
	2	-5.23	R_{42}	+1.18±.01
	3	-6.92	R_{41}	+2.47±.01
	4.	-8.13	R_{40}	+4.28±.01
	5.	-9.05	R_{39}	+6.51±.01
	6	-9.79	R_{38}	+9.13± .01

Table F.1: Adjustment data for LoG X DFG

3. Starting from Adj No.1 :

Set the input voltage (V_{in}) by means of pot.1 as given in table F.1 and observe the output voltage V_0. In case the output voltage is out of tolerance, adjust the appropriate variable resistor as given

under column "Adjust". The adjustments must be completed in order from adjustment no.1 to 6 and after each adjustment, the previous adjustment must be checked.

For example to adjust the positive output DFG set V_{in} to -2.15 volts with coefficient potentiometer 1. Adjust R_{22} until V_o measures +.28 volts. Set V_{in} to -5.23 volts; adjust R_{42} until V_0 measures + 1.18 volts. Recheck adjustment number 1 and reset R_{22} if necessary. Recheck adjustment 2 and reset if necessary. Set adjustment 3; recheck adjustments 1 & 2. The recheck procedure is necessary only for adjustments 1,2 & 3. Adjustments 4,5 & 6 can be made independently.

Results:

The output voltage range of Log X DFG is 0 to 10 for an input voltage range of 0.1V to 10V. From 0.1V to 1 V input, the output is within 2% accuracy. Whereas from 1V to 10V input, the output is well within .5% accuracy. In Fig.5.5 solid line represents the desired LOG X curve, whereas the X marks represent the actual output of the DFG.

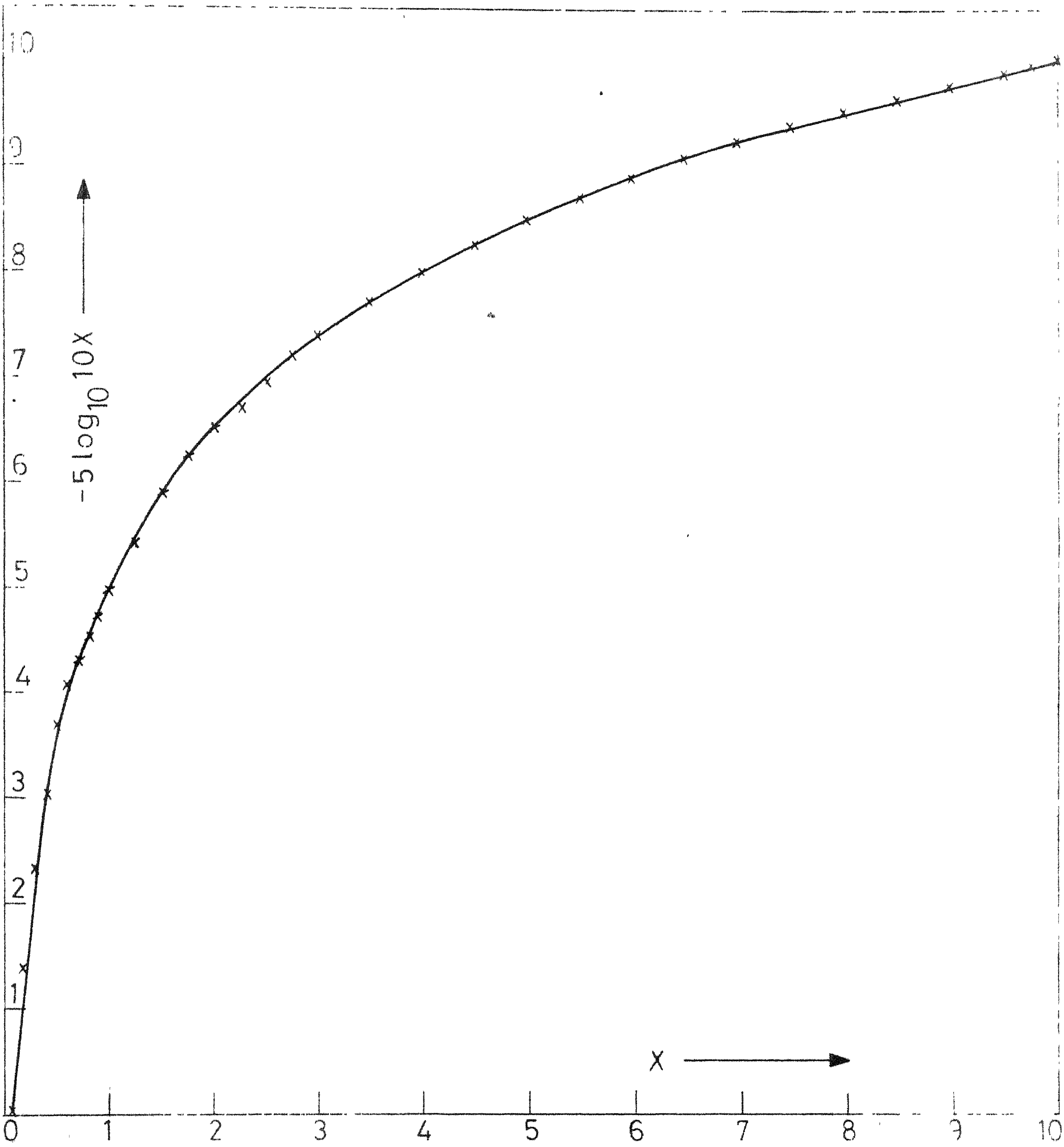


Fig. F-55 Plot of X vs. $-5\log_{10} 10X$ for DFG